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L. C. Pusey

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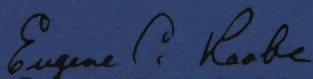
LEXINGTON, MASSACHUSETTS

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FOR THE COMMANDER

Eugene C. Raabe, Lt. Col., USAF
Chief, ESD Lincoln Laboratory Project Office

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

HIGH RESOLUTION SPECTRAL ESTIMATES

L. C. PUSEY

Group 92

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ABSTRACT

The Maximum Likelihood Method and the Maximum Entropy Method of spectral estimation are described and interpreted in terms of the innovation filter concept. Algorithms are developed for mapping an observed data sequence into the spectral estimates. The resolution performance of these spectral estimation algorithms is compared by applying each to simulated data containing two tones in white noise.

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I. INTRODUCTION

A. Problem Statement

We suppose that one observes a segment, $r(t)$ $0 \leq t \leq T$, of a sample function from a zero-mean stationary random process and wishes to generate an estimate of the power density spectrum. When it is desired to distinguish between sharply peaked components of the spectrum at some minimal separation, Δ_f, then the choice of a spectral estimate is largely dependent on the time-bandwidth product, $W = T\Delta f$. If $W >> 1$ then any one of a large number of schemes will achieve the desired resolution with a reasonably small estimate variance. In many situations, however, the observation interval is constrained to be relatively short (e.g. when $r(t)$ may only be considered stationary over a short time interval) and one must select an estimate subject to the requirement, $W \sim 1$.

B. The Conventional Method

When the conventional method (Blackman-Tukey 1959)[11] is employed, the observed data is used to obtain the covariance estimate

$$\tilde{K}(\tau) = (T - \tau)^{-1} \int_{\tau}^{T} r(t)\, r^*(t - \tau)\, dt = \tilde{K}(-\tau) \quad ; \qquad (1.1)$$

$$0 \leq \tau \leq T_m \leq T$$

Where T_m is the maximum lag used. The spectral estimate is the Fourier transform of the product, $\tilde{K}(\tau)\, d(\tau)$, where $d(\tau)$ is a given symetric "lag window" which vanishes for $|\tau| > T_m$.

Since $\tilde{K}(\tau)$ is an unbiased estimate of the covariance, $E\left[r(t)\, r^*(t-\tau)\right]$, the average estimated spectrum is the convolution of the true spectrum with the spectral window,

$$D(f) \overset{\Delta}{=} \int_{-\infty}^{\infty} d(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad .$$

Therefore, our high resolution (small bias) requirement translates into a narrow band constraint on the spectral window.

Unfortunately, the estimate variance is approximately inversely proportional to the spectral window bandwidth. In addition, narrow band windows generally have significant sidelobes which may produce spurious peaks in the estimate. We are thus left with the likelihood that there is no spectral window which provides satisfactory performance for the situation of interest.

C.   The Data Adaptive Methods

Recently, two spectral estimation methods have appeared in the literature which give us a promising alternative approach to this problem. These are the "data adaptive" methods - the M. L. M. (Maximum Likelihood Method)[*] and the M. E. M. (Maximum Entropy Method).[†] Similar to the conventional method, each of the data adaptive methods consists of a procedure for obtaining the spectral estimate from an estimate of the covariance, $K(\tau)$, $|\tau| \leq T_m$.[‡] The new methods, however, relinquish the linearity and simplicity of description associated with the

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*The M. L. M. was developed for application to frequency-wavenumber spectral analysis by Capon, [Reference 6], and applied to time series analysis by Lacoss, [Reference 2].
† The M. E. M. and its application to time series analysis first appeared in a paper by Burg, [Reference 7].
‡ We distinguish here between $K(\tau)$, which is any estimate of the covariance, and the particular estimate, $\tilde{K}(\tau)$.

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conventional method in an attempt to provide improved estimate performance when high resolution is required.

For convenience, we give a short description of the methods here. A more complete description is contained in Chapter II.

The M. L. M. spectral estimate at the frequency f Hz is obtained as the average output power of a particular causal convolutional filter of duration T_m seconds when the input is a random process with covariance $K(\tau)$ for $|\tau| \leq T_m$. The filter is designed to pass the power in a narrow band about f Hz and minimize the power due to other (interfering) spectral components. The average output power of the filter is given by

$$P_L(f: T_m) = \left[\int_0^{T_m} \int_0^{T_m} e^{-j2\pi ft} Q(t, u) e^{j2\pi fu} \, dt du \right]^{-1} \qquad (1.2)$$

where we have assumed the existance of the inverse kernel, $Q(t, u)$, of $K(\tau)$ defined by the integral equation

$$\int_0^{T_m} Q(t, u) K(u - x) \, du = \delta(t - x) \quad ; \quad 0 < t, x < T_m$$

To describe the M. E. M., we assume that the measured covariance is of the form

$$K(\tau) = N_0 \delta(\tau) + \beta(\tau) = K^*(-\tau) ; \quad |\tau| \leq T_m$$

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and consider the causal convolutional filter of duration T_m seconds which is obtained as the solution to the integral equation

$$N_o\, h(\tau: T_m) + \int_0^{T_m} h(u: T_m)\, \beta(\tau - u)\, du = \beta(\tau) \quad ; \quad 0 \le \tau \le T_m \quad (1.3)$$

The M. E. M. spectral estimate is given by the Fourier transform of a certain positive definite extension of the measured covariance. The extension is

$$K_E(\tau) = N_o\, \delta(\tau) + \beta_E(\tau) = K_E^*(-\tau) \quad ; \quad |\tau| < \infty$$

Where $\beta_E(\tau)$ is the solution to the Wiener-Hopf integral equation:

$$N_o\, h(\tau: T_m) + \int_0^{T_m} h(u: T_m)\, \beta_E(\tau - u)\, du = \beta_E(\tau) \quad ; \quad 0 \le \tau < \infty \quad (1.4)$$

It can be shown from 1.4 that the filter, $\delta(\tau) - h(\tau: T_m)$, whitens a process with covariance $K_E(\tau)$ and the spectral height of the white output is N_o. Therefore, the M. E. M. spectral estimate is given by

$$P_E(f: T_m) \overset{\Delta}{=} \int_{-\infty}^{\infty} K_E(\tau)\, e^{-j2\pi f\tau}\, d\tau = N_o \left[1 - \int_{-\infty}^{T_m} h(\tau: T_m)\, e^{-j2\pi f\tau}\, d\tau \right]^{-2} \quad (1.5)$$

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The M. E. M. extension of the measured covariance is developed in Chapter II by using the concept of an "innovation process". Other interpretations* of the extension appear in References 7 and 9.

We note that each of the data adaptive methods attempts to improve on the conventional method in a different manner. The M. L. M. employs a "sliding" bandpass filter which adjusts itself to the random process under consideration in such a way that the spectral estimate at one frequency is, in a sense, least affected by spectral components at other frequencies. The M. E. M. extrapolates the measured covariance beyond the largest available lag in a unique way which depends on the measured covariance.

D.   Performance Considerations

At the present time, a complete evaluation of performance for the data adaptive methods has not been accomplished. In order to render this evaluation, one must necessarily specify a scheme for obtaining the measured covariance from the observed data and compute the bias and variance of the spectral estimate for arbitrary spectra (usually by assuming that the observed process is Gaussian). The complex nature of the new methods has thus far hindered this computation and the current results are restricted to the uninteresting case where $W \gg 1$.†

A partial characterization of performance may be obtained by applying the estimation methods to given covariance functions. That is, we take the measured covariance to be a finite segment of a known covariance function and evaluate the estimate performance in terms of the largest lag used and the covariance parameters. The obvious limitation of this approach is that it yields no information about estimate variance. It does, however, provide information which is roughly equivalent to

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*These employ the discrete form of the M. E. M. which is indicated in Chapter II.
† These results are discussed by Lacoss in Reference 2.

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that afforded by a bias computation and is therefore useful as a basis for comparing the resolution obtainable with the different estimation methods.*

In Chapter III we employ this approach to obtain a measure of resolution for the data adaptive methods. This is done by taking the known covariance to be from a random process which consists of two equal power tones in white noise and computing the minimal signal-to-noise ratio necessary to resolve the tones as a function of the time-bandwidth product. The computation provides a useful standard of comparison for the data adaptive and conventional methods. We note, however, that this measure of resolution is somewhat simplified in that it does not directly relate to the situations where the tones are of unequal power or other (interfering) spectral components are present. We have therefore included for reference a computation of the spectral estimates for the general case where n tones of unequal power are superimposed on white noise.

E.   Implementation of the Data Adaptive Methods

The digital computation of the data adaptive spectra from the observed segment, $r(t), 0 \le t \le T$, is considered in Chapter IV where we use discrete forms of the two methods developed in Chapter II. We indicate alternative schemes for estimating the covariance from the observed data and develop algorithms for mapping the observed data into the spectral estimate.

In Chapter V we employ three such schemes with each of the data adaptive methods and apply the six spectral estimates to data acquired by numerically simulating two tones in white noise. The simulations enable us to provide a qualitative comparison of the estimate performances.

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*Some comparisons of this type have been made by Lacoss [Reference 2] .

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II. THE DATA ADAPTIVE METHODS

We present here both the continuous and discrete forms of the data adaptive methods. For either method, the continuous form consists of a mapping from the measured covariance, $K(\tau)$ $|\tau| \leq T_m$, to the spectral estimate, $P(f : T_m)$ $|f| < \infty$. To guarantee the existence of the map in each case, we shall assume that the measured covariance is a fixed segment of the form

$$K(\tau) = N_0 \, \delta(\tau) + \beta(\tau) = K^*(-\tau); \, |\tau| \leq T_m$$

where N_0 is positive, $\beta(\tau)$ is complex - valued and continuous on $\left[-T_m, T_m\right]$, and $K(\tau)$ is positive definite.*

To develop the discrete form of each method, we suppose that the measured covariance is available as $2N+1$ samples of a continuous positive definite covariance segment on $\left[-T_m, T_m\right]$ at uniform intervals of lag. We denote the samples by the vector $\underline{\Psi} = \text{col}(\Psi(-N), \ldots, \Psi(-1), \Psi(0), \Psi(1), \ldots, \Psi(N))$ where $\Psi(i)$ is the covariance segment evaluated at $i\Delta$, and $N\Delta = T_m$. In this notation, the discrete form is a mapping from $\underline{\Psi}$ to the spectral estimate, $P(f : T_m)$, $|f| < \infty$.

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*The positive definite constraint requires that

$$N_0 \int_0^{T_m} |\varphi(x)|^2 \, dx + \int_0^{T_m} \int_0^{T_m} \varphi^*(u) \, \beta(u-v) \, \varphi(v) \, du \, dv$$

be positive for all square-integrable functions $\varphi(x)$ on $\left[0, T_m\right]$ such that $\int_0^{T_m} |\varphi(x)|^2 \, dx > 0$. Note that $\beta(\tau)$ is not necessarily a covariance function since we have not required that it be positive.

The M. L. M. is described in terms of a classical optimal filtering problem. The two forms of this method are readily obtained by considering continuous and discrete versions of the optimal filter.

The M. E. M. provides a certain positive definite extension of the measured covariance. We develop the extension via the concept of an "innovation process".

A.   The M. L. M.

The M. L. M. spectral estimate at the point f_o Hz , $P_L (f_o:T_m)$, is obtained as the solution of a classical optimal filtering problem. We consider the causal convolutional filter of duration T_m seconds, $h_{f_o} (t)$, which satisfies the constraint

$$H_{f_o} (f_o) = \int_0^{T_m} h_{f_o} (t) \, e^{-j2\pi f_o t} dt = 1$$

and minimizes the average output power when the input is a zero-mean random process whose covariance agrees with the measured covariance on interval $\left[-T_m, T_m \right]$.* The M. L. M. spectral estimate is precisely the average output power of this filter. That is,

$$P_L \left(f_o : T_m \right) = E \left[|x_o (t)|^2 \right] \tag{2.1}$$

Where

$$x_o (t) = \int_0^{T_m} h_{f_o} (\tau) \, x (t - \tau) \, d t \tag{2.2}$$

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*   Since the average output power is independent of the phase of $H_{f_o} (f_o)$, the causal constraint is not essential.

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If the solution for the optimal filter is employed in 2.1 and 2.2 to determine the spectral estimate in terms of the measured covariance, the result is

$$P_L(f_o : T_m) = \left[\int_0^{T_m} \int_0^{T_m} e^{-j2\pi f_o t} \, Q(t,u) \, e^{j2\pi f_o u} \, dt \, du \right]^{-1} \tag{2.3}$$

where $Q(t,u)$ is the inverse kernel of $K(t-u)$ defined by the relation

$$\int_0^{T_m} Q(x,t) \, K(t-u) \, dt = \delta(x-u); \quad 0 < x, \, u < T_m \tag{2.4}$$

By the assumed form of the measured covariance, the inverse kernel is well defined

$$Q(t,u) = (1/N_o) \left[\delta(t-u) - H(t,u) \right]; \quad 0 \le t, \, u \le T_m \tag{2.5}$$

where $H(t,u)$ (the Fredholm resolvent of $\beta(t-u)$ on $[0, T_m]$) is the solution to the Fredholm integral equation

$$N_o H(t,\tau) + \int_0^{T_m} H(t,u) \, \beta(u-\tau) \, du = \beta(t-\tau); \quad 0 \le \tau, \, t \le T_m \tag{2.6}$$

To indicate the discrete form of the M.L.M., we define the $(N+1) \times (N+1)$ measured covariance matrix

$$\left[\Phi_N \right]_{ij} \triangleq \psi(j-i) \qquad 1 \le i, \, j \le N+1 \tag{2.7}$$

and the $N+1$ dimensional sampled frequency vector

$$F_N \triangleq \text{col} \left(1, \, e^{j2\pi f \Delta} \, \dots, \, e^{j2\pi f N \Delta} \right). \tag{2.8}$$

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A discrete replica of the preceding optimal filtering argument yields the analogous result[2]

$$P_L (f:T_m) = \left[\underline{F}_N^T \ \Phi_N^{-1} \ \underline{F}_N^* \right]^{-1} \tag{2.9}$$

We note here that the M. L. M. does not provide an extension of the measured covariance. That is, the inverse Fourier transform of the spectral estimate does not, in general, agree with the measured covariance on the interval $\left[-T_m, \ T_m \right]$.

In the next part, we indicate that the M.L. M. does extend the measured covariance and provide an interpretation of the extension in terms of the innovation filter.

B.   The M. E. M.

Consider a zero-mean stationary random process $x (.)$ on $[0, \ \infty]$ with covariance

$$E\left[x (t) x^* (u) \right] = K_E (t-u) = N_o \delta (t-u) + \beta_E (t-u) ; \quad 0 \le t, \ u < \infty$$

It is desired to select $\beta_E (\tau)$, $|\tau| < \infty$ and a causal convolutional filter $h (\tau : T_m)$ of duration T_m seconds so that the following constraints are satisfied:

1) $\beta_E (\tau) = \beta (\tau) \qquad |\tau| \le T_m$

2)   The process $w (.)$ on $[T_m, \ \infty)$ defined by

$$w (t) \overset{\Delta}{=} x (t) - \int_0^{T_m} h (u : T_m) x (t-u) \, d u ; \qquad T_m \le t < \infty$$

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is a white process for which

$$E [w(t) x^*(u)] = 0 \text{ for } t > u$$

The second constraint implies that

$$E [w(t) x^*(t - \tau)] = K_E(\tau) - \int_0^{T_m} h(u: T_m) K_E(\tau - u) du = 0$$

$$0 < \tau < \infty$$

Consequently, $\beta_E(\tau)$ and $h(\tau : T_m)$ must satisfy the Wiener-Hopf integral equation

$$N_o h(\tau : T_m) + \int_0^{T_m} h(u : T_m) \beta(\tau - u) du = \beta_E(\tau) ; \quad 0 \le \tau < \infty \qquad (2.10)$$

Since $\beta_E(\tau)$, $h(\tau : T_m)$ may be obtained as the solution to the integral equation

$$N_o h(\tau : T_m) + \int_0^{T_m} h(u : T_m) \beta(\tau - u) du = \beta(\tau) ; \quad 0 \le \tau \le T_m \qquad (2.11)$$

Given $h(\tau : T_m)$, it can be shown that 2.10 has a unique positive definite solution, $K_E(\tau)$, and $\beta_E(\tau)^*$ is square-integrable.

Conversely, it can be shown that the functions $\beta_E(\tau)$ and $h(\tau : T_m)$ determined by 2.10 and 2.11 satisfy the above two constraints where the spectral height of $w(t)$ is N_o. The whitening property of the filter $\delta(\tau) - h(\tau : T_m)$ is easily verified by computing $E [w(t)w^*(u)]$ and then employing 2.10.

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* We have obtained these results in a paper which is to be published - "An Innovations Approach to Covariance Extensions."

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The covariance $K_E(\tau)$ is precisely the M.E.M. extension of the measured covariance. Since the filter $\delta(\tau) - h(\tau: T_m)$ whitens the process $x(.)$ in the steady state and the spectral height of $w(.)$ is N_0, the M.E.M. spectral estimate is given by

$$P_E(f: T_m) \triangleq \int_{-\infty}^{\infty} K_E(\tau) e^{-j2\pi f \tau} d\tau =$$

$$N_0 \Big/ \Big| 1 - \int_0 h(\tau: T_m) e^{-j2\pi f \tau} \Big|^2 \qquad (2.12)$$

The existence of the positive definite extension $K_E(\tau)$ depends only on the assumption that $K(\tau)$ is positive definite and is independent of whether $\beta(\tau)$ is positive definite. When the positive definiteness of $\beta(\tau)$ is assumed, it is of interest to inquire whether $\beta_E(\tau)$ is positive definite. It follows from the continuity of the integral term in 2.10 that the difference, $N_0 h(\tau: T_m) - \beta(\tau)$, is continuous for $\tau \geq 0$. When

$$h(T_m: T_m) \triangleq \lim_{\tau \to T_m^-} h(\tau: T_m) \neq 0,$$

then $\beta_E(\tau)$ is discontinuous at the point $\tau = T_m$. Note that $\beta_E(\tau)$ is continuous at the origin by assumption; it then follows that $\beta_E(\tau)$ is not a valid covariance function (see e.g., Papoulis, Reference [12] p. 334). It is easy to demonstrate cases where $\beta(\tau)$ is positive definite and $h(T_m: T_m) \neq 0$ (e.g., if $\beta(\tau) = \sigma > 0$, $|\tau| \leq T_m$, then $h(T_m: T_m) = \sigma / [N_0 + \sigma T_m]$. Consequently, we may conclude that this assumption does not imply that $\beta_E(\tau)$ is positive definite. *

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* If it is known that the observed process consists of a signal process plus an uncorrelated white process, then we have additional information beyond the measured covariance. Therefore, the result does not indicate that the M.E.M. generates an "unreasonable" spectral estimate from the measured covariance.

It is instructive to interpret the M. E. M. extension in terms of the "innovation process of $x(.)$". This process is obtained by driving the causal linear filter $\Omega(\tau: t) = \delta(\tau) - h(\tau: t)$ (the "innovation filter"),[†] with $x(.)$.[‡] That is,

$$v(t) \overset{\Delta}{=} x(t) - \int_0^t h(u:t) \, x(t-u) \, du \; ; \quad 0 \le t < \infty \tag{2.13}$$

Where $h(\tau: t)$ is the solution to the integral equation

$$N_o h(\tau: t) + \int_0^t h(u:t) \, \beta_E(\tau - u) \, du = \beta_E(\tau) \; ; \quad 0 \le \tau \le t < \infty \tag{2.14}$$

It can be verified that the innovation process $v(.)$ is white with spectral height N_o.

To provide a physical interpretation of the innovation process, we employ a result due to T. Kailath [3] which indicates that we may write $x(.)$ as the sum of a signal process $s(.)$ and a correlated white noise process $z(.)$ where the "future" noise is uncorrelated with the "past" signal. That is,

$$x(t) = s(t) + z(t) \; ; \quad 0 \le t < \infty$$

wnere $E[z(t) z^*(u)] = N_o \delta(t-u)$, $E[z(t) s^*(u)] = 0$

for $t > u$ and $\beta_E(t-u) = E[s(t) s^*(u)] + E[s(t) w^*(u)]$ for $t \ge u$.

With this representation of $x(.)$, the least-squares linear estimate of $s(t)$, given the observation $x(u)$ $0 \le u \le t$, is precisely

$$\hat{s}(t) = \int_0^t h(u:t) \, x(t-u) \, du \; ; \quad 0 \le t < \infty$$

[†] For notational convenience, we represent the innovation filter by giving its response at time t due to an impulse applied at time $t - \tau$.

[‡] The observations of this section concerning the innovation process apply more generally to a stationary process $y(.)$ on $(0, \infty)$ with covariance, $c(\tau) = N_o \delta(\tau) + d(\tau)$, where $c(\tau)$ is positive definite.

This indicates that $h(\tau:t)$ may always be regarded as a causal least-squares filter. The innovation process then has the interpretation illustrated in Fig. 2.1.

Since the covariance of $x(.)$ is the M. E. M. extension of $K(\tau)$, the innovation filter is determined for $t \leq T_m$ by the integral equation

$$N_o h(\tau:t) + \int_0^t h(u:t)\,\beta(\tau-u)\,du \;=\; \beta(\tau)\,; \quad 0 \leq \tau \leq T_m \qquad (2.15)$$

For $t > T_m$, the innovation filter is given by *

$$\Omega(\tau:t) \;=\; \Omega(\tau:T_m) \;\overset{\Delta}{=}\; \delta(\tau) - h(\tau:T_m)\,; \quad T_m < t < \infty \qquad (2.16)$$

That is, the innovation filter determined by the M. E. M. covariance $K_E(\tau)$ is time invariant for $t > T_m$. Consequently, the M. E. M. extension of the measured covariance has a particularly simple interpretation in terms of the innovation filter. For $t \leq T_m$, the innovation filter is determined by the measured covariance. For $t > T_m$, the extension is chosen so that the innovation filter is given by 2.16.

This interpretation is readily applied to the discrete form of the M. E. M.. To this end we let $x(i) = 0, 1, 2, \ldots$ be a zero-mean stationary random process with covariance

$$E\,[\,x(i)x^*(j)\,] \;=\; \psi_E(i-j) \qquad\qquad 0 \leq i,\; j < \infty$$

where $\quad \psi_E(i) \;=\; \psi(i) \qquad\qquad\qquad -N \leq i \leq N$

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*   From 2.10 we observe that $h(\tau:T_m)$ is a solution to 2.14 for $t > T_m$. Since the solution is unique, the result follows.

x(t) = s(t) + z(t) →

→

$\hat{s}$(t)

h (τ : t)

$\Omega(\tau$: t)

v(t) →

E [x(t) x*(u)]

= N$_o$ δ(t - u) + β_E (t - u)

E [v(t)v*(u)]
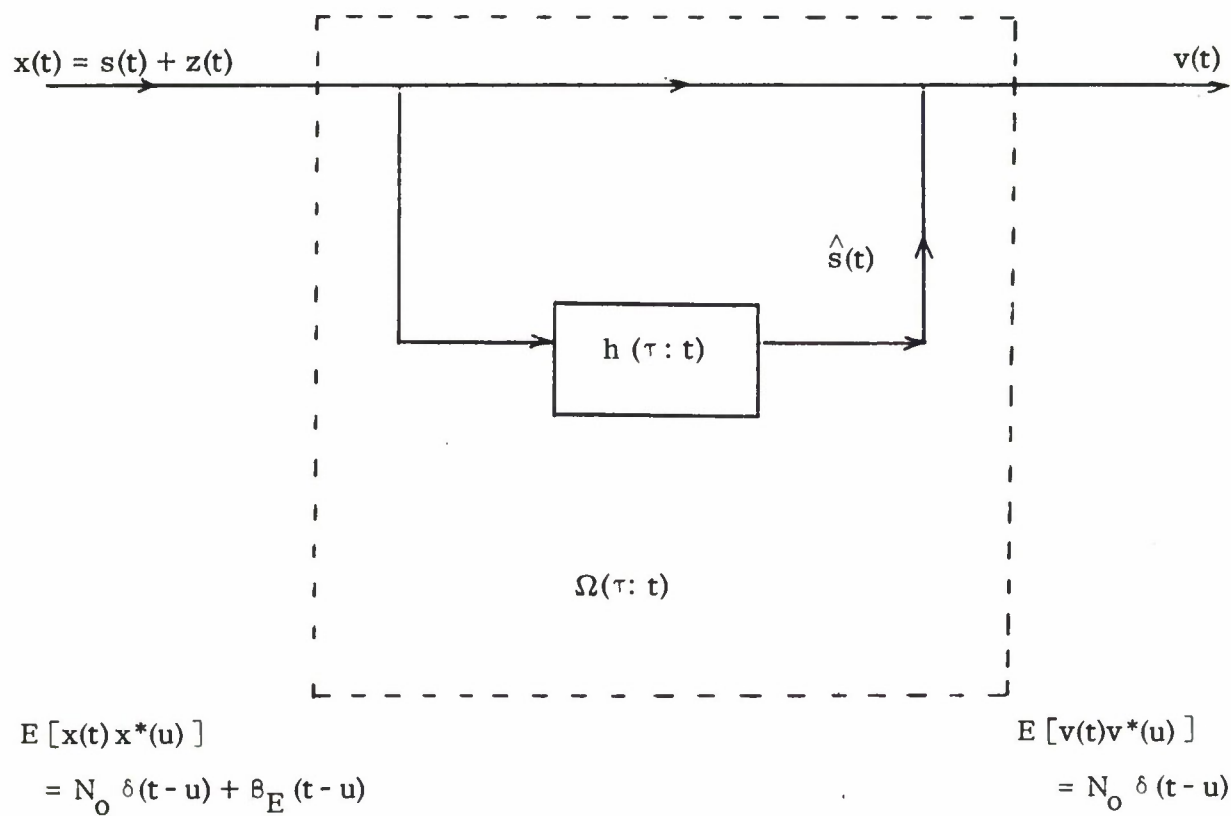
= N$_o$ δ (t - u)

Fig. 2.1 - The Innovation Process.

Let $\hat{x}(n)$ denote the least-squares linear estimate of $x(n)$ given the observations, $x(i)$, $i = 0, 1, \ldots, n-1$, where $\hat{x}(0) = 0$. If we interpret $\hat{x}(n)$ as the output of a causal linear filter $- \Gamma_n(i)$, † the innovation process of $x(.)$ is given by

$$\varepsilon(n) \overset{\Delta}{=} x(n) - \hat{x}(n) =$$

$$x(n) + \sum_{i=1}^{n} \Gamma_n(i) \, x(n-i) = \sum_{i=0}^{n} \Omega_n(i) \, x(n-i) \qquad (2.17)$$

$$n = 0, 1, 2, \ldots$$

where

$$\Omega_n(i) \overset{\Delta}{=} \left\{ \begin{array}{ll} 1 & i = 0 \\ \\ \Gamma_n(i) & 1 \leq i \leq n \end{array} \right\}$$

is the innovation filter.

We observe that the innovation process is just the estimate error as it evolves in time. The filter weights at time n are then chosen so as to minimize the quantity, $p_n \overset{\Delta}{=} E[\varepsilon(n) \, \varepsilon^*(n)]$ (the average output power of the innovation filter at time n). Recall that this is equivalent to requiring that the error be orthogonal to the observations. That is, we require

$$E[\varepsilon(n) \, x^*(n-j)] = 0 \text{ for } j = 1, 2, \ldots, n \qquad (2.18)$$

It follows readily from 2.17 and 2.18 that

$$E[\varepsilon(\ell) \, \varepsilon^*(n)] = p_n \, \delta_{\ell n}$$

so that the innovation process is white as in the continuous case. The variance is, however, time-varying.

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†   The negative sign is chosen by convention.

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Applying 2.18 to 2.17, we obtain $n + 1$ linear equations for p_n and the filter weights at time n:

$$\Psi_E(j) + p_o \, \Gamma_n(j) + \sum_{i=1}^{n} \Psi_E^*(j-1) \, \Gamma_n(i) = 0$$

$$j = 1, 2, \ldots, n \qquad (2.19)$$

$$P_n = P_o + \sum_{i=1}^{n} \Gamma_n(i) \, \Psi_E^*(i)$$

In vector notation these equations become

$$\underline{\Phi}_n \, \Omega_n^* = \underline{P}_n$$

Where $\underline{\Phi}_n$ is the $(n+1) \times (n+1)$ covariance matrix, $\left[\underline{\Phi}_n \right]_{ij} = \qquad (2.20)$

$\Psi_E(i-j)$, Ω_n is the innovation filter at time n,

$$\Omega_n = \begin{bmatrix} 1 \\ \underline{\Gamma}_n \end{bmatrix} \qquad ; \qquad \left[\underline{\Gamma}_n \right]_i = \Gamma_n(i),$$

and $\underline{P}_n = \text{col}(p_n, 0, \text{---}, 0)$

Analogous to the continuous case, we may describe the M. E. M. extension of the measured covariance in terms of the innovation filter. For $0 \le n \le N$, the innovation filter and its average output power are determined from the measured covariance by 2.20. For $N + 1 \le \infty$, the covariance extension is chosen so that

$$\underline{\Omega}_n(i) = \begin{cases} \Omega_N(i) & 0 \le i \le N \\ & \qquad ; \; N + 1 \le n < \infty \\ 0 & N + 1 \le i \le n \end{cases}$$

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Applying this condition to 2.19 we see that the extension is determined by the recursive relation

$$\Psi_E(j) = \Psi_E^*(-j) = -\sum_{i=1}^{N} \Psi_E(j-i)\,\Gamma_n(i) \qquad (2.21)$$

$$j = N+1,\ N+2,\ \ldots$$

It can be shown that the sequence obtained from 2.21 is square-summable and positive definite when Φ_N is positive definite.[*] It follows that the discrete Fourier transform is well defined by the limit in the mean

$$d_{\Psi_E}(\lambda) \overset{\Delta}{=} \text{L.i.m.}_{n \to \infty} \sum_{k=-n}^{n} \Psi_E(k)\, e^{-j2\pi k\lambda}$$

Since the convolutional filter Ω_N whitens the process $x(.)$ in the steady state and its average output power is p_N, we obtain

$$d_{\Psi_E}(\lambda)\,|d_{\Omega_N}(\lambda)|^2 = p_N \qquad (2.22)$$

where

$$d_{\Omega_N}(\lambda) = 1 + \sum_{k=1}^{N} \Gamma_N(k)\, e^{-j2\pi k\lambda}$$

The M. E. M. spectral estimate is then given by

$$P_E(f:T_m) = \Delta d_{\Psi_E}(\Delta f) = \Delta P_N \Big/ |d_{\Omega_N}(\Delta f)|^2$$

Employing the sampled frequency vector defined in Part A, this may be written as

$$P_E(f:T_m) = \Delta P_N / |\underline{F}_N^T\ \underline{\Omega}_N^*|^2 \qquad (2.23)$$

---

[*]   The sequence is said to be positive definite if Φ_n is a positive definite matrix for $n = 1, 2, \ldots$

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Consequently, the M. E. M. spectral estimate is obtained from the discrete measured covariance by determining the innovation filter and its average output power at time N from 2.20. In the next part we indicate that the M. L. M. spectral estimate may be obtained from the innovation filter and its average output power for times $1 \leq n \leq N$, and demonstrate a relationship between the M. E. M. and M. L. M.

C.   A Deterministic Relationship between the Data Adaptive Methods

In the last part, we demonstrated that the innovation filter and its average output power at time $1 \leq n \leq N$ is determined from the measured covariance by the vector equation.

$$\underline{\sigma}_n \, \Omega_n^* \;=\; \underline{P}_n \qquad\qquad 1 \leq n \leq N \qquad\qquad (2.24)$$

Our first step here is to write the M. L. M. spectral estimate in terms of the innovation filter.

Let $x(i)$ $i = 0, 1, 2, \ldots, N$ be a zero-mean random process with covariance

$$E\,[x(i)\,x^*(j)] \;=\; \Psi(i-j) \qquad 0 \leq i, \; j \leq N$$

and define the vector

$$\underline{x}_N \;=\; \mathrm{col}\,(x(N), \; x(N-1), \ldots, \; x(0))$$

If $\underline{G}_N$ denotes the $(N+1) \times (N+1)$ matrix,

$$\underline{G}_N \;\triangleq\;$$

$\underline{\Omega}^T{}_N$			
0	$\underline{\Omega}^T{}_{N-1}$		
0	0	$\underline{\Omega}^T{}_{N-2}$	
0	0		01

$$(2.25)$$

then the innovation process of x (.) may be written in the vector form

$$\underline{\mathcal{E}}_N = \underline{G}_N \underline{X}_N \tag{2.26}$$

where $\quad \underline{\mathcal{E}}_N = \text{col} (\ (n),\ \mathcal{E}(n-1),\ -\ -\ -\ \mathcal{E}(0)\ )$

Noting that

$$\underline{\mathcal{E}}_N = E\left[\underline{X}_N \underline{X}_N{}^{*T}\right] \text{ and } \underline{D}_N = E\left[\underline{\mathcal{E}}_N \underline{\mathcal{E}}_N{}^{*T}\right]$$

where $\underline{D}_N$ is the diagonal matrix

$$\underline{D}_N \overset{\Delta}{=} \begin{bmatrix} p_N & & \underline{0} \\ & & \\ & & \\ \underline{0} & & p_0 \end{bmatrix}$$

we have from 2.26

$$\underline{D}_N = \underline{G}_N \underline{\Phi}_N \underline{G}_N^{*\ T} \tag{2.27}$$

It follows that the inverse measured covariance matrix is given by

$$\underline{\Phi}_N{}^{-1} = \underline{G}_N^{*\ T} \underline{D}_N{}^{-1} \underline{G}_N \tag{2.28}$$

From 2.9 we obtain the M. L. M. spectral estimate

$$P_L{}^{-1} (f: N\Delta) = \underline{F}_N{}^T \underline{\Phi}_N{}^{-1} F_N{}^* = \underline{F}_N{}^T \underline{G}_N^{*\ T} \underline{D}_N{}^{-1} \underline{G}_N F_N{}^*$$

$$= 1/p_0 + |\Omega_1^T \underline{F}_1^*|^2 / p_1 + \ldots + |\Omega_N^T \underline{F}_N^*|^2 / p_N \tag{2.29}$$

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By equation 2.23 we may write

$$P_E(f: i\Delta) = \Delta P_i \Big/ \left| \Omega_i^T F_{-i}^* \right|^2 \qquad\qquad i = 0, 1, 2, \ldots, N$$

where $P_E(f: 0) \stackrel{\Delta}{=} \Delta P_0$.

It then follows from 2.29 that

$$P_L^{-1}(f: N\Delta) = \Delta \sum_{i=0}^{N} P_E^{-1}(f: i\Delta) \qquad\qquad (2.30)$$

That is, the reciprocal of the M. L. M. spectrum is obtained by averaging the recipro-cals of the M. E. M. spectra obtained as the maximum lag of the measured covariance varies from zero to $N\Delta$.* This result is still subject to interpretation. However, it does suggest that the M. L. M. spectral estimate has a lower resolution capability and is more stable then the M. E. M. estimate.

Equation 2.30 suggests that a differential relationship exists between the continuous forms of the data adaptive estimates. We now show that

$$\frac{d}{dT} P_L^{-1}(f: T) = P_E^{-1}(f: T) \qquad\qquad (2.31)$$

We employ 2.3 and 2.5 to write the M. L. M. spectral estimate as

$$P_L^{-1}(f: T) = (1/N_0)\left[T - \int_0^T \int_0^T e^{-j2\pi f t} H(t, u: T) e^{j2\pi f u} \, dt \, du \right] \qquad (2.32)$$

---

*   This result recently appeared in a paper by J. Burg, Reference 4.

Where $H(t, u: T)$ is the Fredholm resolvent of $\beta(t-u)$ on $[0, T]$, as determined by the integral equation

$$N_0 \, H(t, \tau: T) + \int_0^T H(t, u: T) \, \beta(u-\tau) \, du = \beta(t-\tau) \, ; \quad 0 \le \tau, t \le T \qquad (2.33)$$

Differentiating 2.32, we obtain

$$N_0 \frac{d}{dT} P_L^{-1}(f: T) = 1 - \int_0^T e^{-j2\pi fT} H(T, u: T) \, e^{j2\pi fu} \, du$$

$$- \int_0^T e^{-j2\pi ft} H(t, T: T) \, e^{j2\pi fT} \, dt$$

$$- \int_0^T \int_0^T e^{-j2\pi ft} \frac{\partial}{\partial T} H(t, u: T) \, e^{j2\pi fu} \, dt \, du \qquad (2.34)$$

To evaluate the derivative indicated in the last term of 2.34, we use the resolvent identity (see e.g. R. Bellman, Reference 5)

$$\frac{\partial}{\partial T} H(t, u: T) = -H^*(T, t: T) \, H(T, u: T)$$

It then follows easily from the conjugate symmetry of the Freedholm resolve and a change of variable that 2.34 may be written as

$$\frac{d}{dT} P_L^{-1}(f: T) = \left| 1 - \int_0^T H(T, T-\tau: T) \, e^{-j2\pi f} \, dt \right|^2 \Big/ N_0$$

Since $H(T, T-\tau: T) = h(\tau: T)$ (Compare 2.11 and 2.33), 2.31 follows from 2.12.

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D.   Summary

In this chapter we indicated the continuous and discrete forms of the data adaptive methods and provided an interpretation of the M. E. M. extension in terms of the innovation filter. We have also shown that the discrete time M. L. M. spectrum may be obtained from the innovation filter and its average output power. This result is gainfully employed in Chapter IV where we consider a scheme for implementing the two methods in which the innovation filter is estimated directly from the observed data.

The continuous forms are employed in the next chapter to obtain an analytical comparison of the resolution capabilities of the two data adaptive methods and the conventional (Blackman - Tukey) method when the measured covariance consists of a pair of tones in white noise.

III.   A MEASURE OF RESOLUTION FOR THE DATA ADAPTIVE METHODS

As indicated in the introduction, some measure of estimate performance may be obtained by applying the estimation methods to known covariance functions. In this chapter we consider the case where the given covariance function is from a random process which consists of two pure tones in additive white noise. Our objective in applying the estimation methods to this simple model is to provide a quantitative measure of resolution which may serve as a basis for comparison of the two data adaptive methods and the conventional method. For simplicity, we consider the situation where the tones have equal power so that the measured covariance is of the form.

$$K(\tau) = N_0 \delta(\tau) + \sigma^2 e^{j 2\pi f_1 \tau} + \sigma^2 e^{j 2\pi f_2 \tau}; \quad |\tau| \leq T_m$$

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$$P_c \, (f{:}\, T_m) \;=\; D\,(f) * \left[\, N_0 + \sigma^2 \, \delta\,(f - f_1) + \sigma^2 \, \delta\,(f - f_2)\,\right]$$

$$= \; N_0 \; d\,(0) + \sigma^2 \; D\,(f - f_1) + \sigma^2 \; D\,(f - f_2)$$

$$= \; N_0 \left[\, 1 + \theta/\, T_m \, \left(D\,(f - f_1) + D\,(f - f_2) \right) \,\right] \tag{3.1}$$

From 3.1, we obtain

$$R_c \, (\theta,\, W) \;=\; \left[\, 1 + (\theta/T_m) \left(D\,(0) + D\,(W/T_m) \right) \right] \Big/ \left[\, 1 + (2\,\theta/T_m) \; D\,(W/2\, T_m) \,\right]$$

The locus of points θ, W for which the tones are just separated is obtained by setting $R_c \, (\theta,\, W)$ equal to one. This yields the equation

$$D\,(0) + D\,(W/\, T_m) \;=\; 2\, D\,(W/2T_m) \tag{3.2}$$

Equation 3.2 determines a "cutoff" time-bandwidth product W_c which depends on the lag window used. For $W > W_c$ the tones are separated and for $W \le W_c$ the tones are not separated. We have illustrated this in Figure 3.1 by plotting $\theta'(W)$ as a vertical line centered at W_c. The values of W_c for two typical lag windows are shown in the same figure.

Note that this result is apparent from the second term in 3.1 since the white noise level appears simply as an additive constant in the spectral estimate. We shall obtain a quite differenct result for the data adaptive methods.
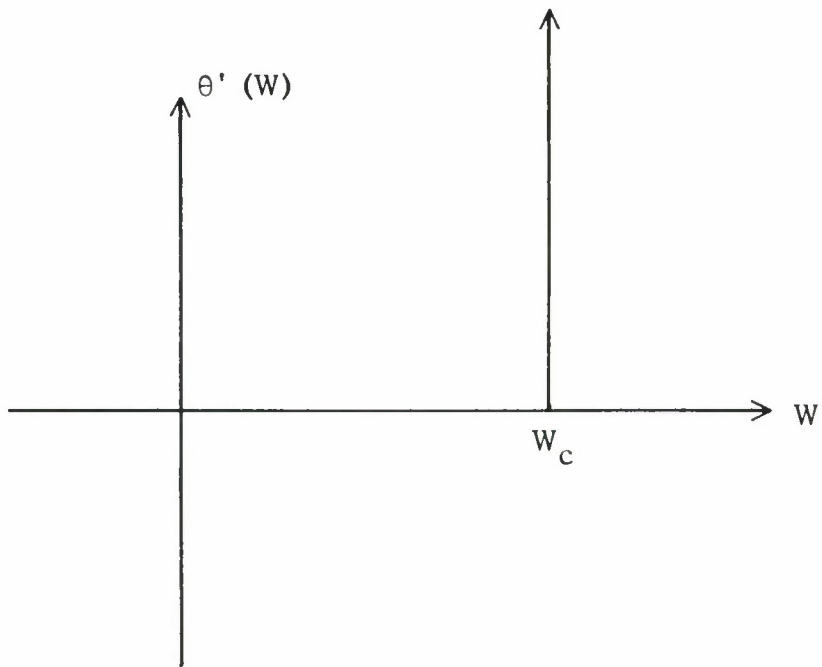
lag window	W_c
rectangular: $d(\tau) = 1 \qquad |\tau| \le T_m$	.681
triangular $d(\tau) = \left(1 - \dfrac{|\tau|}{T_m}\right)|\tau| \le T_m$	.874

Fig. 3.1 - θ' (W) for the Conventional Method.

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B.   The M. E. M.

We consider first the more general case where the measured covariance is

$$K(\tau) = N_0 \delta(\tau) + \sum_{i=1}^{n} \sigma_i^2 e^{j2\pi f_i \tau} \quad ; \quad |\tau| \leq T_m$$

The M. E. M. spectral estimate is obtained from 2.11 and 2.12. For the above measured covariance, the former equation becomes

$$N_0 h(\tau: T_m) + \sum_{i=1}^{n} \sigma_i^2 e^{j2\pi f_i \tau} \int_0^{T_m} h(u: T_m) e^{-j2\pi f i u} du$$

$$= \sum_{i=1}^{n} \sigma_i^2 e^{j2\pi f i \tau}; \quad 0 \leq \tau \leq T_m \tag{3.3}$$

We let $H(f: T_m)$ denote the Fourier transform of $h(\tau: T_m)$. Multiplying both sides of 3.3 by $e^{-j2\pi f \tau}$ and integrating, we obtain the equation

$$H(f: T_m) = \sum_{i=1}^{n} \theta_i \left[1 - H(f_i: T_m) \right] e^{j\pi(f_i - f) T_m} \operatorname{sinc}\left(\pi(f_i - f) T_m\right) \tag{3.4}$$

Where $\theta_i \overset{\Delta}{=} \sigma_i^2 T_m / N_0$ is the signal-to-noise ratio corresponding to the ith tone and the function sinc (X) is defined by

$$\operatorname{sinc}(X) \overset{\Delta}{=} \frac{\sin(X)}{X}$$

We note that 3.4 is a representation of $H(f: T_m)$ in terms of its samples at the n tone frequencies. To determine the samples, we evaluate 3.4 at $f = f_j$ for

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j = 1, 2, . . . , n. This results in n linear equations for

$H(f_j : T_m)$ j = 1, 2, . . . , n:

$$H(f_j : T_m) = \sum_{i=1}^{n} \theta_i \left[1 - H(f_i : T_m) \right] e^{j\pi(f_i - f_j)} \text{ sinc } \left[\pi(f_i - f_j) T_m \right] \qquad (3.5)$$

$$j = 1, 2, \ldots, n$$

Therefore, the M. E. M. spectral estimate for the general n-tone case may be obtained from 3.5, 3.4, and 2.12.

When n = 2 and θ_1 = θ_2 = θ, 3.5 has the solution

$$1 - H(f_1 : T_m) = 1 - H^*(f_2 : T_m) =$$

$$\left(1 + \theta - \theta e^{-j\pi W} \text{ sinc } (\pi W) \right) \Big/ \left((1 + \theta)^2 - \theta^2 \text{ sinc}^2 (\pi W) \right) \qquad (3.6)$$

Equation 3.4 becomes

$$H(f : T_m) = \theta \sum_{i=1}^{2} \left[1 - H(f_i : T_m) \right] e^{-j 2\pi(f_i - f) T_m} \cdot \left[\text{sinc} (\pi(f_i - f) T_m) \right] \qquad (3.7)$$

It follows from 2.12 that

$$R_E(\theta, W) = \left| 1 - H\left(\frac{f_1 + f_2}{2} : T_m \right) \right|^2 \Big/ \left| 1 - H(f_1 : T_m) \right|^2 \qquad (3.8)$$

With some algebraic manipulation 3.8 reduces to

$$R_E(\theta, W) = [1 + \theta - \theta \text{ sinc } (\pi W)]^4 / B \qquad (3.9)$$

where $B = (1 + \theta)^2 - 2(1 + \theta) \theta \text{sinc } (\pi W) \cos (\pi W) + \theta^2 \text{ sinc}^2 (\pi W)$.

Setting this term equal to one, we obtain a cubic equation for θ' (W) :

$$\alpha^4 \theta^3 + 4 \alpha^3 \theta^2 + \left[5 \alpha^2 - 2 (1 - \alpha) \mathcal{F} \right] \theta + 2\alpha - 2(1 - \alpha) \mathcal{F} = 0 \qquad (3.10)$$

where $\alpha \overset{\Delta}{=} 1 - \text{sinc } (\pi W), \mathcal{F} = 1 - \cos (\pi W)$.

This equation may be solved numerically by using, e.g., Newton's method. In Figure 3.2 we have plotted the solution on a dB scale ($20 \log_{10} (.)$) for relatively small values of the time-bandwidth product. For larger values of W the M. E. M. has the same resolution as the covential method when the rectangular window is used in that it separates the tones for any positive signal-to-noise ration if $W > W_c \simeq .681$.

The plot indicates an important distinction between the conventional method and the M. E. M. . For the former method the tones cannot be separated when $W \leq W_c$. When the M. E. M. is employed, however, the tones may, in principle be resolved for any non-zero time-bandwidth product if the signal-to-noise ration is sufficiently large. [*]

We now show that similar results hold for the M. L. M.

C. The M. L. M.

To obtain the M. L. M. spectral estimate for the general n-tone case, we employ 2.4. This yields the equation

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[*]   Bear in mind that this result is obtained by using a known covariance function. When real data is used to generate the estimate, the variance and bias of the estimate may preclude effective resolution of the tones when the time-bandwidth product is near zero.
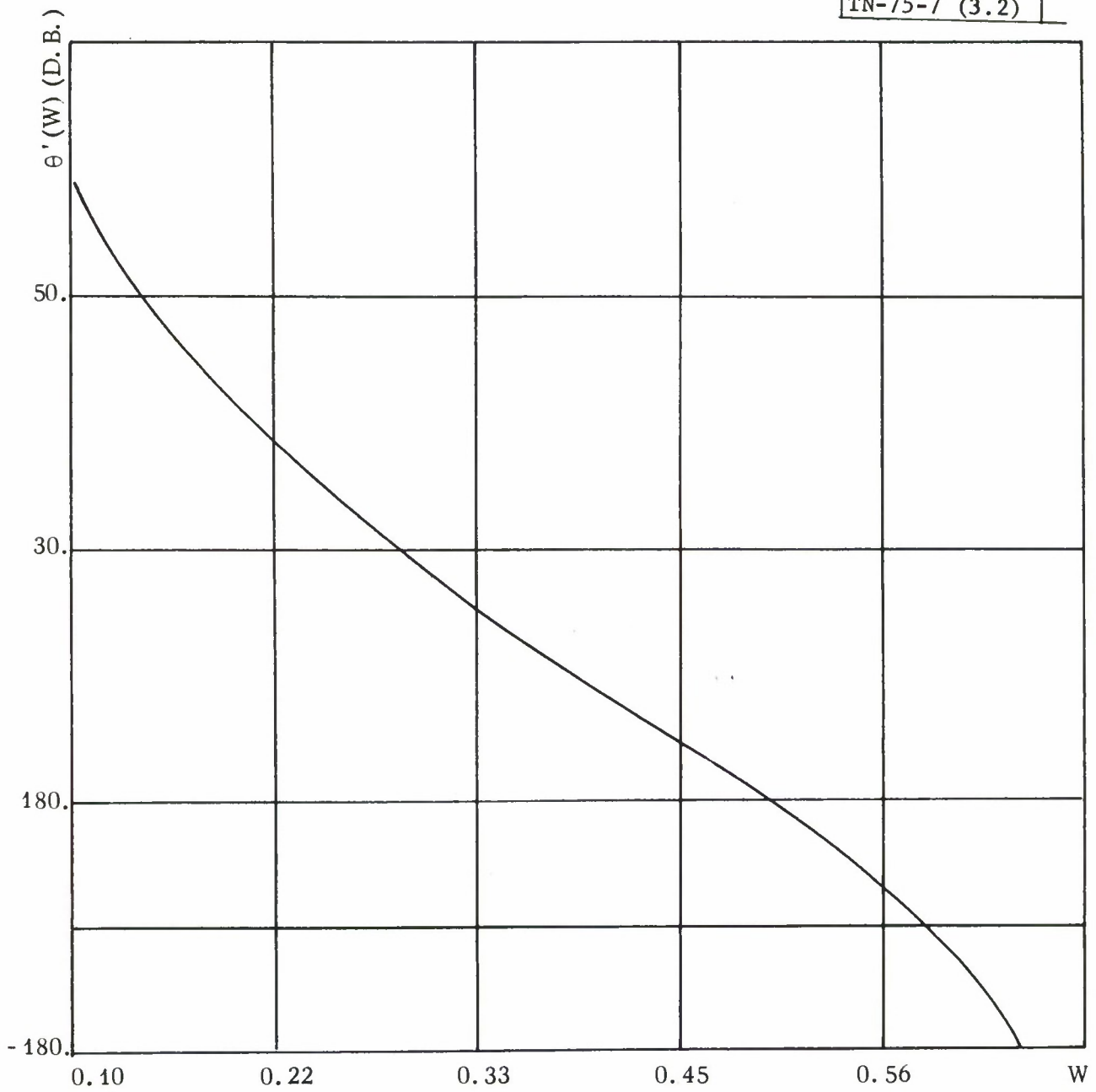
Fig. 3.2 - θ'(W) for The M. E. M.

$$\int_0^{T_m} Q(x, t) \left[N_0 \, \delta(t - u) + \sum_{i=1}^n \sigma_i^2 \, e^{j2\pi f_i(t - u)} \right] dt$$

$$= N_0 \, Q(x, u) + \sum_{i=1}^n \sigma_i^2 \int_0^{T_m} Q(x,t) \, e^{-j2\pi f_i(t - u)} dt$$

$$= \delta(x - u) \tag{3.11}$$

Define the function

$$S(f, f') \overset{\Delta}{=} \int_0^{T_m} \int_0^{T_m} e^{-j2\pi fx} \, Q(x, u) \, e^{j2\pi f' u} \, dx \, du \tag{3.12}$$

We multiply both sides of 3.11 by $e^{-j2\pi fx} \, e^{-j2\pi f' u}$ and integrate:

$$S(f, f') + \sum_{i=1}^n \theta_i \, e^{-j\pi(f_i - f')T_m} \, \mathrm{sinc}\left[\pi(f_i - f') T_m \right] S(f, f_i)$$

$$= \left(T_m/N_0 \right) e^{-j\pi(f - f')T_m} \, \mathrm{sinc}\left[\pi(f - f') T_m \right] \tag{3.13}$$

This equation gives $S(f, f')$ in terms of the samples, $S(f, f_i)$ $i = 1, 2, \ldots, n$. Evaluating 3.13 at $f = f_j$ $j = 1, 2, \ldots, n$, we obtain n linear equations for the samples:

$$S(f, f_j) = \sum_{i=1}^n \theta_i \, e^{-j\pi(f_i - f_j)T_m} \, \mathrm{sinc}\left[\pi(f_i - f_j) T_m \right] S(f, f_i)$$

$$= \left(T_m/N_0 \right) e^{-j\pi(f - f_j) T_m} \, \mathrm{sinc}\left[\pi(f - f_j) T_m \right] \qquad j = 1, 2, \ldots, n \tag{3.14}$$

We note from 2.3 that the M. L. M. spectral estimate is

$$P_L(t: T_m) = \left[S(f, f) \right]^{-1} \tag{3.15}$$

Consequently, 3.14 and 3.13 determine the estimate for the n-tone case.

When n = 2 and $\theta_1 = \theta_2 = \theta$, 3.14 has the solution for $S(f, f_1)$

$$S(f, f_1) = (T_m/N_0) \, e^{-j\pi(f - f_1) \, T_m} \left\{ (\theta+1) \, \text{sinc} \left[\pi(f - f_1) \, T_m \right] \right.$$

$$\left. -\theta \, \text{sinc} \left(\pi W \, \text{sinc} \left[\pi(f - f_2) \, T_m \right] \right) \right\} \Big/ \left\{ (\theta + 1)^2 - \theta^2 \, \text{sinc}^2 (\pi W) \right\} \tag{3.16}$$

$S(f, f_2)$ is obtained by interchanging f_1 and f_2 in 3.16.

Equation 3.13 then gives the reciprocal of the M. L. M. estimate in terms of the samples:

$$S(f, f) = (T_m/N_0) - \theta \sum_{i=1}^{2} e^{-j\pi(f_i - f) \, T_m} \, \text{sinc} \left[\pi(f_i - f) \, t_m \right] S(f, f_i) \tag{3.17}$$

We note that

$$R_L(\theta, W) = S \left[(f_1 + f_2)/2, \, (f_1 + f_2)/2 \right] \Big/ S(f_1, f_2) \tag{3.18}$$

With some algebra 3.18 reduces to

$$R_L(\theta, W) = \left\{ (\theta+1)^2 - \theta^2 \, \text{sinc}^2 (\pi W) - 2\theta \, \text{sinc}^2 (\pi W/2) \cdot \right.$$

$$\left. \left[1 + \theta \left(1 - \text{sinc} (\pi W) \right) \right] \right\} \Big/ \left\{ 1 + \theta \left(1 - \text{sinc}^2 (\pi W) \right) \right\} \tag{3.19}$$

Setting $R_L(\theta, W)$ equal to one, we obtain the solution for $\theta'(W)$

$$\theta'(W) = \left\{ 2 \, \text{sinc}^2 (\pi W/2) - \text{sinc}^2 (\pi W) - 1 \right\} \Big/$$

$$\left\{ 1 - \text{sinc}^2 (\pi W) - 2 \, \text{sinc}^2 (\pi W/2) \left[1 - \text{sinc} (\pi W) \right] \right\} \tag{3.20}$$

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θ'(W) D. B.

Fig. 3.3 - The M.L.M.

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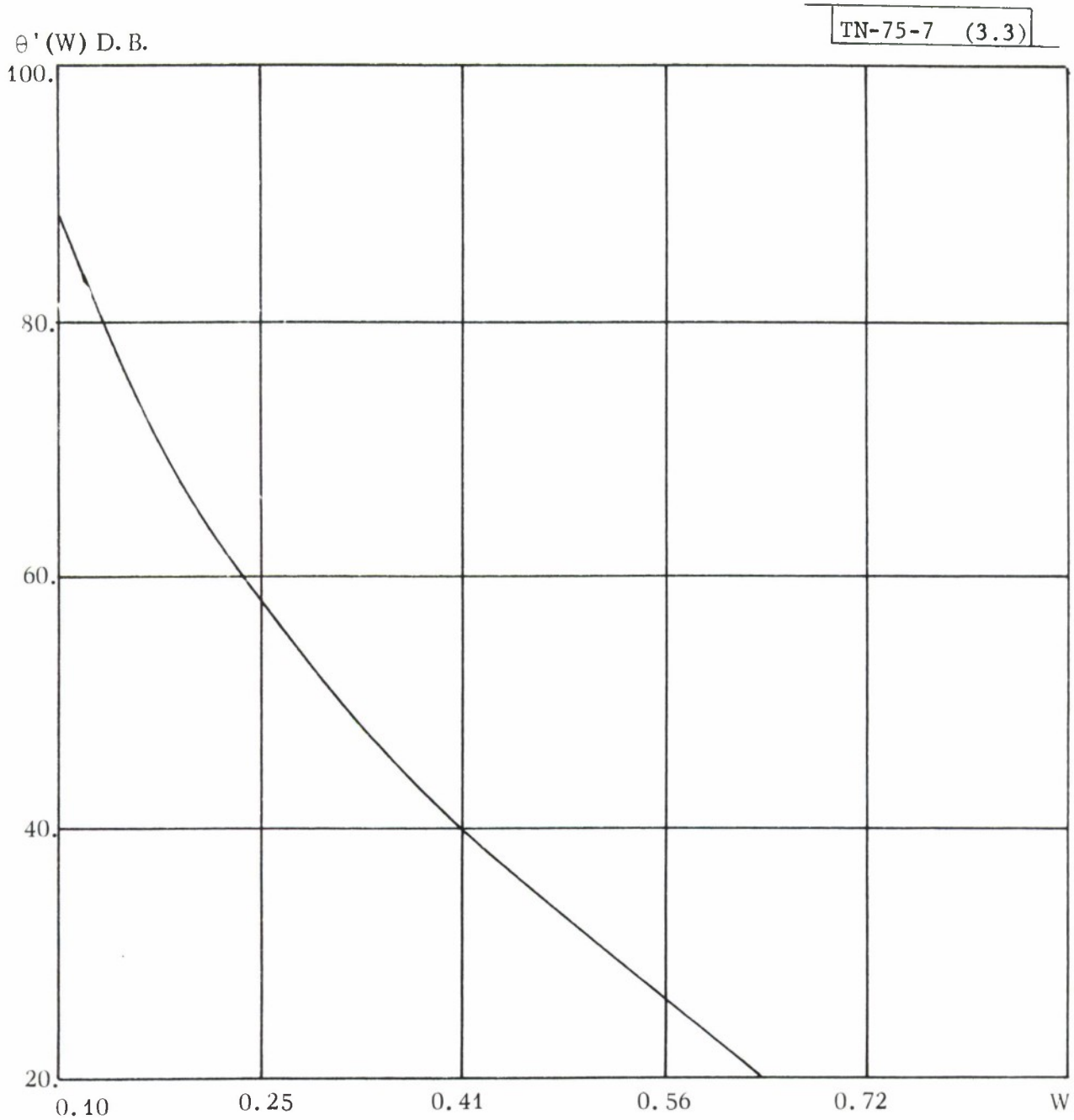
This function is plotted in Figure 3.3 over the same range of time-bandwidth product considered for the M. E. M. For larger values of W the resolution of the M. L. M. is identical to that of the conventional triangular window in that the M. L. M. separates the tones for any positive signal-to-noise ratio if $W > W_c \cong .874$.

We note that the M. L. M. has the same general behavior as the M. E. M. in that we can theoretically resolve tones which are arbitrarily close together if the signal to-noise-ratio is sufficiently large. Comparing figures 3.2 and 3.3, we see that the M. E. M. requires less signal-to-noise ratio to separate the tones than the M. L. M. The difference is about 20 dB over the range of W considered.

IV.   IMPLEMENTATION OF THE DATA ADAPTIVE METHODS

In Chapter II we indicated the mapping from the measured covariance, $\Psi(i)$ $i = -N, \ldots, 0 \ldots, N$, to the innovation filter and its average output power:

$$\underline{\Phi}_n \, \underline{\Omega}_n^* = \underline{P}_n \qquad\qquad n = 0, 1, \ldots, N \qquad (4.1)$$

where

$$\left[\underline{\Phi}_n \right]_{ij} = \Psi(j - i) \qquad\qquad i, j = 1, \ldots, n+1$$

$$\underline{P}_n = \mathrm{col}\,(p_n, 0 \ldots, 0)$$

It was shown that both the M. E. M. and M. L. M. spectra could be written in terms of $\underline{\Omega}_n$, p_n $n = 0, 1, \ldots, N$:

$$P_E(f: T_m) = \Delta p_N / | \underline{F}_N^T \, \underline{\Omega}_N^* |^2 \qquad\qquad (4.2)$$

$$P_L(f: T_m) = \left[1/p_0 + | \underline{F}_1^T \, \underline{\Omega}_1^* |^2 / p_1 + - - - + | \underline{F}_N^T \, \underline{\Omega}_N^* |^2 / p_N \right]^{-1} \quad (4.3)$$

In this chapter, we consider schemes for obtaining the spectra from the observed data segment, $r(t)$ $0 \leq t \leq T$. We assume that $r(t)$ is sampled uniformly at the M points, $i \Delta$ $i = 0, 1, \ldots, M - 1$, where $(M - 1) \Delta = T$, $N < M$ and denote the samples by the vector

$$\underline{r} = \text{col} (r_0, r_1, \ldots, r_{M-1}).$$

By 4.2 and 4.3, our problem becomes one of estimating the innovation filter and its average output power from the sampled data. We put forth two general schemes for this purpose (Parts B and C). With the first (indirect scheme), we compute the measured covariance from the sampled data and solve 4.1 for $\underline{\Omega}_n$, p_n $n = 0, 1, \ldots, N$. The second (direct scheme) constitutes a procedure for estimating $\underline{\Omega}_n$, p_n $n = 0, 1, \ldots, N$ directly from the sampled data.[*] This scheme is based on the fact that the innovation filter weights are chosen so as to minimize the average output power (see Part B, Chapter II).

Both schemes employ recursive relations for $\underline{\Omega}_n$ and p_n which we now develop.

A. Recursive Equations for the Innovation Filter

We assume throughtout this part that the measured covariance is given and $\underline{\Phi}_n$ is positive definite. As in Part B of Chapter II, we denote the innovation filter at time n by

$$\underline{\gamma}_n = \begin{bmatrix} 1 \\ \hline \underline{\Gamma}_n \end{bmatrix}$$

Where $\underline{\Gamma}_n$ is the least-squares filter.

To write 4.1 in terms of $\underline{\Gamma}_n$, we define the n-dimensional vector

$$\underline{\psi}_n = \begin{bmatrix} \psi(1) \\ \psi(2) \\ \vdots \\ \psi(n) \end{bmatrix} \tag{4.4}$$

---

[*] We have worked out the details of a scheme suggested by Burg, Reference 8.

In this notation, 4.1 becomes

$$\underline{\Phi}_{n-1} \underline{\Gamma}_n^* = -\underline{\psi}_n^* \qquad\qquad n = 1, 2, \ldots, N \qquad (4.5)$$

$$p_n = p_0 + \underline{\psi}_n^T \underline{\Gamma}_n^* \qquad\qquad n = 1, 2, \ldots, N \qquad (4.6)$$

where $p_0 = \psi(0)$

For $\underline{Y} = \mathrm{col}\,(Y_1, Y_2, \ldots, Y_n)$, define the inversion of $\underline{Y}$ to be the n-dimensional vector $\underline{Y}$ given by

$$\overset{\wedge}{\underline{Y}} \overset{\Delta}{=} \mathrm{col}\,(Y_n, Y_{n-1}, \ldots, Y_1) \qquad (4.7)$$

Employing this definition and 4.5, we may write the equations for $\underline{\Gamma}_{n+1}$ in block form

$$\begin{bmatrix} \underline{\Phi}_{n-1} & \vdots & \overset{\wedge}{\underline{\psi}}_n \\ \cdots & \vdots & \cdots \\ \overset{\wedge}{\underline{\psi}}_n^{*\,T} & \vdots & p_0 \end{bmatrix} \begin{bmatrix} \underline{B}_n^* \\ \cdots \\ \gamma_{n+1}^* \end{bmatrix} = - \begin{bmatrix} \underline{\psi}_n^* \\ \cdots \\ \psi_{(n+1)}^* \end{bmatrix} \qquad (4.8)$$

Where

$$\underline{B}_n \overset{\Delta}{=} \mathrm{col}\,(\Gamma_{n+1}(1), \ldots, \Gamma_{n+1}(n))$$

$$\gamma_{n+1} \overset{\Delta}{=} \Gamma_{n+1}(n+1)$$

Therefore, we have

$$\underline{\Phi}_{n-1} \underline{B}_n^* + \overset{\wedge}{\underline{\psi}}_n \gamma_{n+1}^* = -\underline{\psi}_n^* \qquad (4.9)$$

Since $\underline{\Phi}_{n-1}$ is a Toeplitz matrix, it follows readily from 4.5 that

$$-\overset{\wedge}{\underline{\psi}}_n = \underline{\Phi}_{n-1} \overset{\wedge}{\underline{\Gamma}}_n \qquad (4.10)$$

Employing 4.5 and 4.10 in 4.9, we obtain

$$\underline{\Phi}_{n-1} \underline{B}_n^* = \underline{\Phi}_{n-1} \underline{\Gamma}_n^* + \underline{\Phi}_{n-1} \overset{\wedge}{\underline{\Gamma}}_n \gamma_{n+1} \qquad (4.11)$$

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Since $\underline{\Phi}_{n-1}$ is non-singular, 4.11 yields the result

$$\underline{B}_n = \underline{\Gamma}_n + \hat{\underline{\Gamma}}_n^* \ \gamma_{n+1} \tag{4.12}$$

or

$$\underline{\Gamma}_{n+1} = \left[\begin{array}{c} \dfrac{\underline{\Gamma}_n + \hat{\underline{\Gamma}}_n^* \ \gamma_{n+1}}{\gamma_{n+1}} \end{array} \right] \tag{4.13}$$

This equation is the desired recursive relation for the innovation filter. It gives the least-squares filter at time $n+1$ in terms of γ_{n+1} and the least-squares filter at time n.

We now develop a recursive relation for the average output power and indicate an algorithm for obtaining the innovation filter and its average output power from the measured covariance.

From 4.8 we also have the scalar equation

$$\hat{\underline{\Psi}}_n^{*T} \underline{B}_n^* + p_0 \ \gamma_{n+1}^* = -\Psi^*(n+1) \tag{4.14}$$

We employ 4.12 and 4.6 in 4.14 to obtain the equation

$$\gamma_{n+1} = -\left[\hat{\underline{\Psi}}_n^{T} \underline{\Gamma}_n + \Psi(n+1) \right] \Big/ p_n \tag{4.15}$$

It follows from 4.6 that

$$p_{n+1} = p_0 + \underline{\Psi}_{n+1}^{T} \ \underline{\Gamma}_{n+1}^* =$$

$$p_0 + \underline{\Psi}_n^{T} \underline{B}_n^* + \gamma_{n+1}^* \ \Psi(n+1) \tag{4.16}$$

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By employing 4.12 and 4.15 in 4.16, we obtain a recursive relation for the average output power:

$$p_{n+1} = p_n (1 - |\gamma_{n+1}|^2) \tag{4.17}$$

Note that $p_n > 0 \quad n = 0, 1, \ldots, N$, since $\underline{\Phi}_N$ is positive definite. Consequently, 4.17 implies that $|\gamma_n| \overset{\Delta}{=} |\Gamma_n(n)| < 1 \quad n = 1, 2, \ldots, N$.

Equations 4.13, 4.15 and 4.17 constitute an algorithm for mapping the measured covariance into the innovation filter and its average output power:

1) $p_0 = \psi(0) \quad \underline{\Omega}_0 = [1]$

2) Given $\underline{\Omega}_n$ and p_n, we compute γ_{n+1} from 4.15, Ω_{n+1} and p_{n+1} are then obtained from 4.13 and 4.17, respectively.

B. The Indirect Scheme

To implement the indirect scheme, we obtain the measured covariance as

$$\psi(i) = \frac{1}{M} \sum_{j=i}^{M-1} r_j r_{j-1}^* \qquad i = 0, 1, \ldots, N; \quad N < M$$

The innovation filter and its average output power are then computed via the above algorithm.

We note that the measured covariance is a triangular weighting of the unbiased covariance estimate.

$$\widetilde{K}(i \Delta) = \frac{1}{M-i} \sum_{j=i}^{M-1} r_j r_{j-1}^* \qquad i = 0, 1, \ldots, N; \quad N < M$$

Other weighting is possible, including the rectangular weighting (i.e. $\psi(i) = \widetilde{K}(i \Delta)$). The triangular weighting, however, guarantees that the measured covariance is non-negative definite (see, e.q.,Jenkins and Watts, Reference 10). This is obviously a necessary requirement for the data adaptive methods.

C.   The Direct Scheme

A basic quantity in the recursive equations of Part A is $\gamma_n \overset{\Delta}{=} \Gamma_n(n)$, the response of the innovation filter at time $n \geq 1$ due to a discrete impulse applied at time zero. Since stationary filters are typically represented by this response, we shall refer to the sequence, $\left\{ \gamma_1, \gamma_2, \ldots, \gamma_N \right\}$ as the <u>stationary response</u> of the innovation filter.

In Part A we observed that when $\underline{\Phi}_N$ is positive definite, then the stationary response at time n satisfies the constraint

$$|\gamma_n| < 1 \qquad n = 1, 2, \ldots, N \qquad\qquad (4.18)$$

It is of interest that the converse of this statement also holds. That is, when p_0 is a given positive number and $\left\{ \gamma_1, \gamma_2, \ldots, \gamma_N \right\}$ is a given collection of complex numbers which satisfy 4.18, then there exists a unique positive definite Toeplitz matrix, $\underline{\Phi}_N$, which satisfies 4.1 where $\underline{\Omega}_n$, p_n $n = 0, 1, \ldots, N$ are obtained from $\left\{ \gamma_1, \gamma_2, \ldots, \gamma_N \right\}$ via the recursive equations, 4.13 and 4.17.[*] We note from 4.15 that the elements of this matrix $\psi(j-i) = \left[\underline{\Phi}_N \right]_{ij}$, are given by the recursive equation

$$\psi(n+1) = -\left[p_n \gamma_{n+1} + \underline{\hat{\psi}}_n^T \underline{\Gamma}_n \right] \qquad n = 0, 1, \ldots, N-1$$

where $\psi(0) = p_0$.

Consequently, any estimate, $\hat{p}_0$, $\left\{ \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_N \right\}$ for which $\hat{p}_0$ is positive and 4.18 holds corresponds to a positive definite measured covariance. There is, of course, no need to actually compute the measured covariance when the stationary response is obtained from the sampled data since we may determine the innovation filter and its average output power directly from 4.13 and 4.17.

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[*]   J. Burg suggested this result in Reference 8. We have resently demonstrated it in a paper which is to be published - "An Innovations Approach to Covariance Estensions."

We now estimate the stationary response via a recursive scheme suggested by Burg. To demonstrate the scheme, we let $\underline{\Omega}_{k+1}$ be the innovation filter at time $k+1$ determined by $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_k, \gamma_{k+1}\}$ where $\{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_k\}$ is the estimated stationary response up to time k and γ_{k+1} is a complex parameter (for $k = 0$, we take $\underline{\Omega}_1 = \text{col}(1, \gamma_1)$).

Our first step is to estimate the average output power of $\underline{\Omega}_{k+1}$. We let $\underline{R}_j$ denote the $k+2$ data points

$$\underline{R}_j \overset{\Delta}{=} \text{col}(r_j, r_{j-1}, \ldots, r_{j-(k+1)}); \quad k+1 \le j \le M-1$$

and consider the power estimate

$$\widetilde{P}_{k+1} \overset{\Delta}{=} \left(1 / 2(M-k-1)\right) \sum_{j=k+1}^{M-1} \left\{ |\underline{\Omega}_{k+1}^T \underline{R}_j|^2 + |\hat{\underline{\Omega}}_{k+1}^{*T} \underline{R}_j|^2 \right\} \qquad (4.19)$$

The first term under the summation sign is obtained by convolving $\underline{\Omega}_{k+1}$ with the forward data sequence $\{r_0, r_1, \ldots, r_{M-1}\}$ and the second term results from the covolution of $\underline{\Omega}_{k+1}^*$ with the reverse data sequence $\{r_{M-1}, r_{M-2}, \ldots, r_0\}$.

It is reasonable to take the estimate, $\hat{\gamma}_{k+1}$, to be the value to γ_{k+1} which minimizes P_{k+1}. To determine $\hat{\gamma}_{k+1}$, we use 4.13 to partition $\underline{\Omega}_{k+1}$:

$$\underline{\Omega}_{k+1} = \begin{bmatrix} 1 \\ \hline \underline{\Gamma}_k + \hat{\underline{\Gamma}}_k^* \gamma_{k+1} \\ \hline \gamma_{k+1} \end{bmatrix}$$

$\underline{R}_j$ is partitioned as

$$\underline{R}_j = \begin{bmatrix} X_j \\ \hline Y_j \\ \hline Z_j \end{bmatrix}$$

where X_j and Z_j are scalars.

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With this notation, we may rewrite 4.19 in terms of γ_{k+1}:

$$2(M-K-1)\widetilde{P}_{k+1} = \sum_{j=k+1}^{M-1} \left\{ |x_j + \underline{\Gamma}_k^T \underline{y}_j + \hat{\underline{\Gamma}}_k^{*T} \gamma_{k+1} + \gamma_{k+1} z_j|^2 \right.$$

$$\left. + |\gamma_{k+1}^* x_j + \hat{\underline{\Gamma}}_k^{*T} \underline{y}_j + \underline{\Gamma}_k^T \underline{y}_j \gamma_{k+1}^* + z_j|^2 \right\} \tag{4.20}$$

It is easily verified with elementary calculus that the value of γ_{k+1} which minimizes this expression is given by

$$\hat{\gamma}_{k+1} = -2 \sum_{j=k+1}^{M-1} (X_j + \underline{\Gamma}_k^T \underline{y}_j)(z_j^* + \Gamma_k^T y_j^*) \Bigg/$$

$$\sum_{j=k+1}^{M-1} \left\{ z_j^* + \underline{\Gamma}_k^T \underline{y}_j^{*2} + X_j + \underline{\Gamma}_k^T y_j^2 \right\} \tag{4.21}$$

This may be written in the form

$$\hat{\gamma}_{k+1} = -2 \sum_{j=k+1}^{M-1} Z_k(j) Y_k^*(j) \Bigg/ \sum_{j=k+1}^{M-1} \left\{ |Z_k(j)|^2 + |Y_k(j)|^2 \right\} \tag{4.22}$$

Where

$$Z_k(j) \overset{\Delta}{=} X_j + \underline{\Gamma}_k^T \underline{y}_j, \quad Y_k(j) \overset{\Delta}{=} z_j + \hat{\underline{\Gamma}}_k^{*T} \underline{y}_j$$

Observe that $Z_k(j)$ is the output at time j when $\underline{\Omega}_k$ is convolved with the data sequence:

$$Z_k(j) = r_j + \sum_{\ell=1}^{k} \Gamma_k(\ell) r_{j-\ell} \tag{4.23}$$

and $Y_k(j)$ is the output at time $j-1$ when $\hat{\Omega}_k^*$ is convolved with the data sequence:

$$Y_k(j) = r_{j-(k+1)} + \sum_{\ell=1}^{k} \Gamma_k^*(\ell) r_{j-(k+1)+\ell} \tag{4.24}$$

From 4.22 we obtain the bound

$$|\hat{\gamma}_{k+1}| \leq 2 \sum_{j=k+1}^{M-1} |Z_k(j)| \ |Y_k(j)| \bigg/ \sum_{j=k+1}^{M-1} \left\{ |Z_k(j)|^2 + |Y_k(j)|^2 \right\} \qquad (4.25)$$

Since

$$0 \leq \sum_{j=k+1}^{M-1} \left(|Z_k(j)| - |Y_k(j)| \right)^2 =$$

$$\sum_{j=k+1}^{M-1} \left\{ |Z_k(j)|^2 + |Y_k(j)|^2 \right\} - 2 \sum_{j=k+1}^{M-1} |Z_k(j)| \ |Y_k(j)|$$

it follows from 4.25 that

$$\gamma_{k+1} \leq 1$$

Consequently, $\hat{\gamma}_{k+1}$ is a valid estimate of the stationary response at time $k+1$.

Since $\underline{\Gamma}_k$ is obtained by applying 4.13 to the estimate $(\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_k)$ for $k \geq 1$, equation 4.22 constitutes a recursive scheme for estimating the stationary response of the innovation filter. From 4.21, the starting value for the scheme is

$$\hat{\gamma}_1 = -2 \sum_{j=1}^{M-1} \Gamma_j \Gamma_{j-1}^* \bigg/ \sum_{j=1}^{M-1} \left\{ r_j^2 + r_{j-1}^2 \right\} \qquad (4.26)$$

It is apparent that, if we implement 4.22 by actually computing the convolution of $\underline{\Omega}_k$ with the data sequence, then the number of complex multiply and add operations is prohibitive for large k. Fortunately, this procedure is not necessary as we now indicate.

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By 4.23 and 4.24, we may write

$$Z_{k+1}(j) = r_j + \sum_{\ell=1}^{k+1} \Gamma_{k+1}(\ell)\, r_{j-\ell} = \underline{\Omega}_{k+1}^T\, \underline{R}_j\, Y_{k+1}(j)$$

$$Y_{k+1}(j) = r_{j-(k+2)} + \sum_{\ell=1}^{k+1} \Gamma_{j-(k+2)+\ell} = \Omega_{k+1}^{*T}\, \underline{R}_{j-1}$$

Employing 4.13 and the partition of $\underline{R}_j$ defined previously, we have

$$\underline{\Omega}_{k+1}^T\, \underline{R}_j = X_j + \underline{\Gamma}_k^T\, \underline{y}_j + \hat{\underline{\Gamma}}_k^{*T}\, \underline{y}_j\, \gamma_{k+1} + z_j\, \gamma_{k+1} = Z_k(j) + \gamma_{k+1}\, Y_k(j)$$

and

$$\hat{\Omega}_k^{*T}\, \underline{R}_{j-1} = x_{j-1}\, \gamma_{k+1}^* + \underline{\Gamma}_k^T\, \underline{y}_{j-1}\, \gamma_{k+1}^* + \hat{\underline{\Gamma}}_k^{*T}\, y_{j-1} + z_{j-1}$$

$$= \gamma_{k+1}^*\, Z_k(j-1) + Y_k(j-1)$$

Therefore, we have the recursive equations

$$Z_{k+1}(j) = Z_k(j) + \gamma_{k+1}\, Y_k(j)\,; \quad k+2 \le j \le M-1 \tag{4.27 a}$$

$$Y_{k+1}(j) = \gamma_{k+1}^*\, Z_k(j-1) + Y_k(j-1)\,; \quad k+2 \le j \le M-1 \tag{4.27 b}$$

It follows from 4.26 that, if we intialize 4.27 by defining

$$Z_0(j) \stackrel{\Delta}{=} r_j\,; \quad 1 \le j \le M-1$$

$$Y_0(j) \stackrel{\Delta}{=} r_j\,; \quad 1 \le j \le M-1$$

then equations 4.13, 4.22 and 4.27 constitute an algorithm for mapping the data sequence into the stationary response and the innovation filter.

To complete our description of the direct scheme, we need only indicate how the average output power of the innovation filter is estimated.

$$\text{The logical choice for } \hat{p}_0 = 1/M \sum_{j=0}^{M-1} |r_j|^2 \qquad\qquad (4.28)$$

The remainder of the output power sequence is obtained via 4.17

$$\hat{p}_{k+1} \quad \hat{p}_k (1 - |\hat{\gamma}_{k+1}|^2) \qquad ; \qquad k = 0, 1, \ldots, N-1$$

D.   Summary

In this chapter we developed two basic schemes for estimating the innovation filter and its average output power. With the first (indirect scheme) we solve the system of equations, 4.1, where the measured covariance is taken to be a weighted average of the sampled data. An algorithm for solving this system of equations is indicated in Part A. With the second (direct scheme) we estimate the stationary response of the innovation filter, $\gamma_n = \Gamma_n (n)$, n = 1, 2, . . . , N, directly from the sampled data and determine the innovation filter via equation 4.13. The average output is then computed from equation 4.18 where p_0 (the covariance for zero lag) is estimated by averaging the sampled data.

Variations of these two schemes may be obtained by simply weighting (windowing) the respective estimates. This corresponds to weighting the measured covariance when the indirect scheme is used and weighting the estimated stationary response when the direct scheme is used. We note that the estimated stationary response may be weighted in two different ways:

1) Standard Weighting: $\tilde{\gamma}_k = w(k) \hat{\gamma}_k$ where $\left\{ \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_N \right\}$ is computed via the direct scheme.

2) Iterative Weighting: $\tilde{\gamma}_k = w(k) \hat{\gamma}_k$ where $\hat{\gamma}_k$ is computed at each iteration by

using the <u>weighted</u> stationary response, $\left\{ \tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_{k-1} \right\}$, to determine $\underline{\Gamma}_{k-1}$ in equation 4.21.

Either way yields another valid estimate of the stationary response when $|w(k)| \le 1$, $k = 1, 2, \ldots, N$.

In the next chapter we consider six spectral estimates obtained by employing the indirect scheme, the direct scheme and a weighted version of the direct scheme with each of the data adaptive methods. A qualitative evaluation of the relative performance is conducted by applying each of the estimates to data acquired by simulating a pair of tones in white noise.

V. APPLICATION OF THE SPECTRAL ESTIMATES TO A SIMULATED RANDOM PROCESS

In the previous chapter we developed the indirect and the direct schemes for estimating the innovation filter and its average output power. It was indicated that variations in the latter scheme could be obtained by weighting the estimated stationary response of the innovation filter.

We consider here the six spectral estimates determined by employing each of the following schemes with the M. L. M. and the M. E. M.

1) Indirect

2) Direct

3) Direct-Hanning: This scheme uses an iterative weighting of the estimated stationary response (see Part D, Chapter IV)

The weighting is the Hanning window:

$w(k) = 1/2 \left[1 + \cos(\pi k/M) \right]$ $k = 1, 2, \ldots, N$

where M is the number of data samples and N is the largest time for which the innovation filter is estimated.[*]

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[*] We have chosen w (k) somewhat arbirarily to determine the estimate behavior when a typical iterative weighting is used.

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The spectral estimates are applied to data acquired by numerically simulating two unit power tones in Gaussian white noise. Our model for the sampled data is the following:

$$r_n = e^{j2\pi f_1 n\Delta} + e^{j2\pi f_2 n\Delta} + V_n$$

$$V_n = V_n^R + J V_n^I; \qquad n = 0, 1, 2, \ldots, M-1$$

The 2 M random variables, $\left(V_n^R, V_n^I \right)$ $n = 0, 1, \ldots, M-1$, are independent zero-mean Gaussian with variance $N_0/ 2 \Delta$ where N_0 is the white noise level.

The parameters, (M, Δ, f_1, f_2), are fixed at the values $M = 500$, $\Delta = .01$ sec., $f_1 = 10 H_z.$, $f_2 = 10.5 H_z.$

In Figures 5.1 - 5.48 we have plotted the six estimates on a dB scale $(10 \log_{10} (.))$ for a fixed data sequence and for various values of the parameters, $\theta' \triangleq M \Delta/ N_0$ and $\rho \triangleq N/ M$. Figure 5.0 is the index to these plots. The plots of the spectral estimates obtained with the Direct-Hanning scheme are not included for $\rho < .4$ since these are nearly identical with the corresponding Direct scheme plots.

The spectral plots enable us to provide a qualitative comparison of the estimate performances. The primary measures of performance are the estimate stability as evidenced by the relative "smoothness" of the estimated noise spectrum (ideally the noise spectrum is flat) and the estimate resolution or ability to adequately depict the presence of the two tones.

We observe that the spectral plots support the results of Chapter III in that the estimate resolution improves with increasing signal-to-noise ratio, $\theta = \rho \theta'$. Note, however, that the estimate stability decreases with increasing ρ since the number of data samples is fixed. This is most apparent with the M. E. M. - Direct estimate which is highly unstable for $\rho = .5$ (see Figs. 5.22 and 5.46).

We have drawn the following preliminary conclusions concerning the relative behavior of the six estimates:

1) The Direct and Direct-Hanning Schemes give generally improved performance over the Indirect Scheme for both data adaptive methods. The plots indicate that these schemes yield higher resolution than the Indirect Scheme (see eq. Figs. 5.3 - 5.4 and Figs. 5.15 - 5.16). The stability of the three schemes is roughly the same except for the case where the M. E. M. is used and $\rho > 0.3$. In this case the Indirect and Direct-Hanning schemes have comparable stability (see Figs. 5.21 and 5.24).

2) The M. L. M. - Direct and M. L. M. Direct-Hanning estimates do not differ significantly.

3) For $\rho > .3$ the M. E. M. - Direct-Hanning estimate performs better than the M. E. M. - Direct estimate. This results from the instability of the M. E. M. - Direct estimate for $\rho > .3$ which is apparently reduced significantly by weighting the estimated stationary response (compare Figs. 5.22 and 5.24 and Figs. 5.46 and 5.48). Note that the M. E. M. - Direct estimate is relatively stable for $.2 \leq \rho \leq .3$ and the resolution is not appreciably less than that obtained for $\rho > .3$.

4) The performance of the M. E. M. - Direct for $.2 \leq \rho \leq .3$ is comparable to the performance of the M. L. M. - Direct for $.4 \leq \rho \leq .5$ (Compare eq. Figs. 5.10 and 5.16).

The above conclusions and the additional consideration of computational speed suggest that a reasonable spectral estimation procedure is to employ the M. E. M. - Direct (or the M. E. M. - Direct-Hanning) estimate for $.2 \leq \rho \leq .3$. We emphasize, however, that these are preliminary conclusions and additional testing is clearly required. In particular, other weightings of the stationary response should be considered and the estimates should be applied to a wider range of simulated spectra.

Parameters - $\theta' = M\Delta/N_0 = 5/N_0$, $\rho = N/M = N/500$

$\theta' = 500$		
M. L. M. - Indirect, Direct $\rho = .1, .2, .3, .4, .5$	figs. 5.1 - 5.10	
M. L. M. - Direct-Hanning $\rho = .4, .5$	figs. 5.11 - 5.12	
M. E. M. - Indirect, Direct $\rho = .1, .2, .3, .4, .5$	figs. 5.13 - 5.22	
M. E. M. - Direct-Hanning $\rho = .4, .5$	figs. 5.23 - 5.24	
$\theta' = 50$		
M. L. M. - Indirect, Direct $\rho = .1, .2, .3, .4, .5$	figs. 5.25 - 5.34	
M. L. M. - Direct-Hanning $\rho = .4, .5$	figs. 5.35 - 5.36	
M.E. M. - Indirect, Direct $\rho = .1, .2, .3, .4, .5$	figs. 5.37 - 5.46	
M. E. M. - Direct-Hanning $\rho = .4, .5$	figs. 5.47 -5.48	

Fig. 5.0 -   Index to the Spectral Plots.

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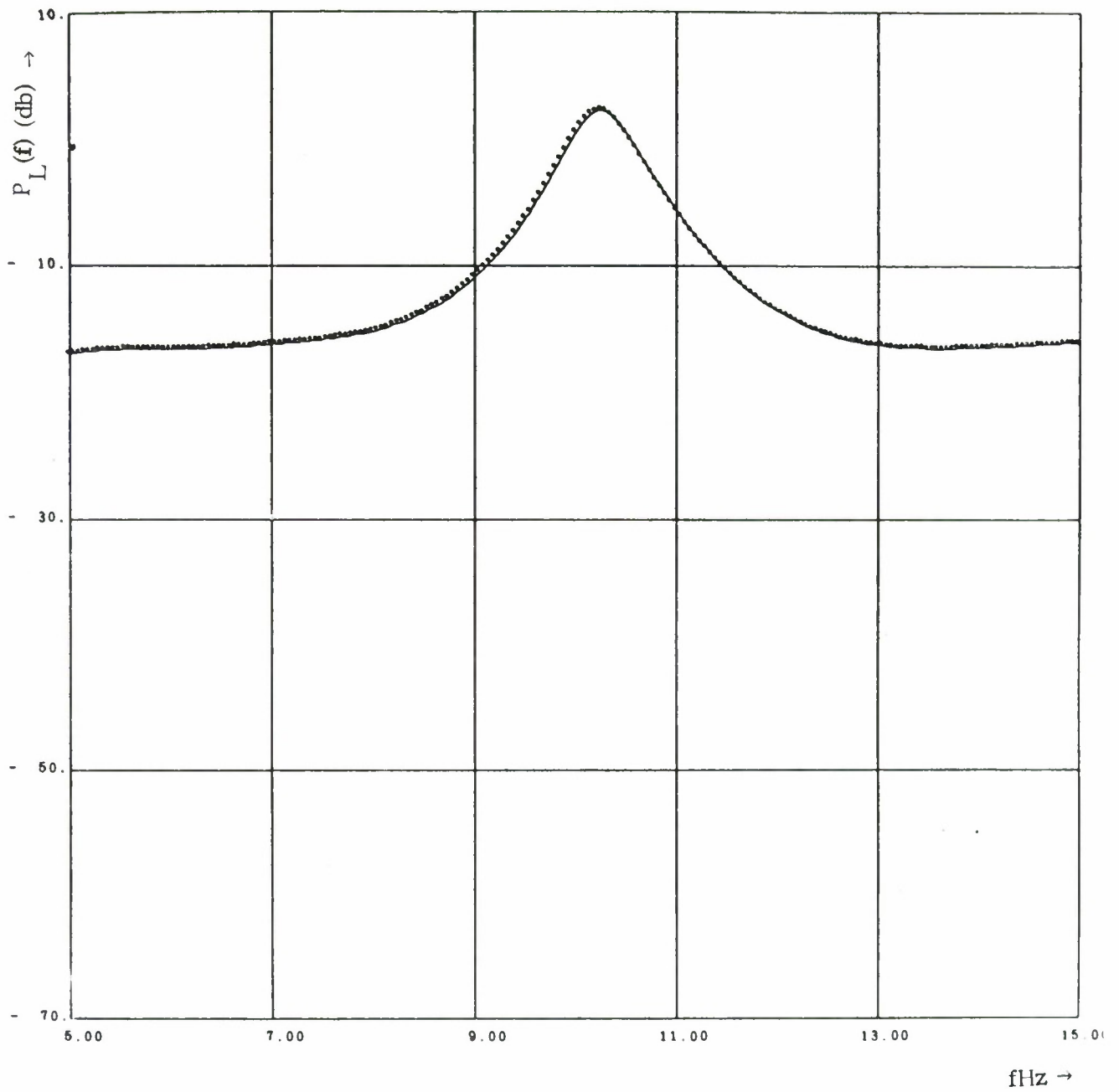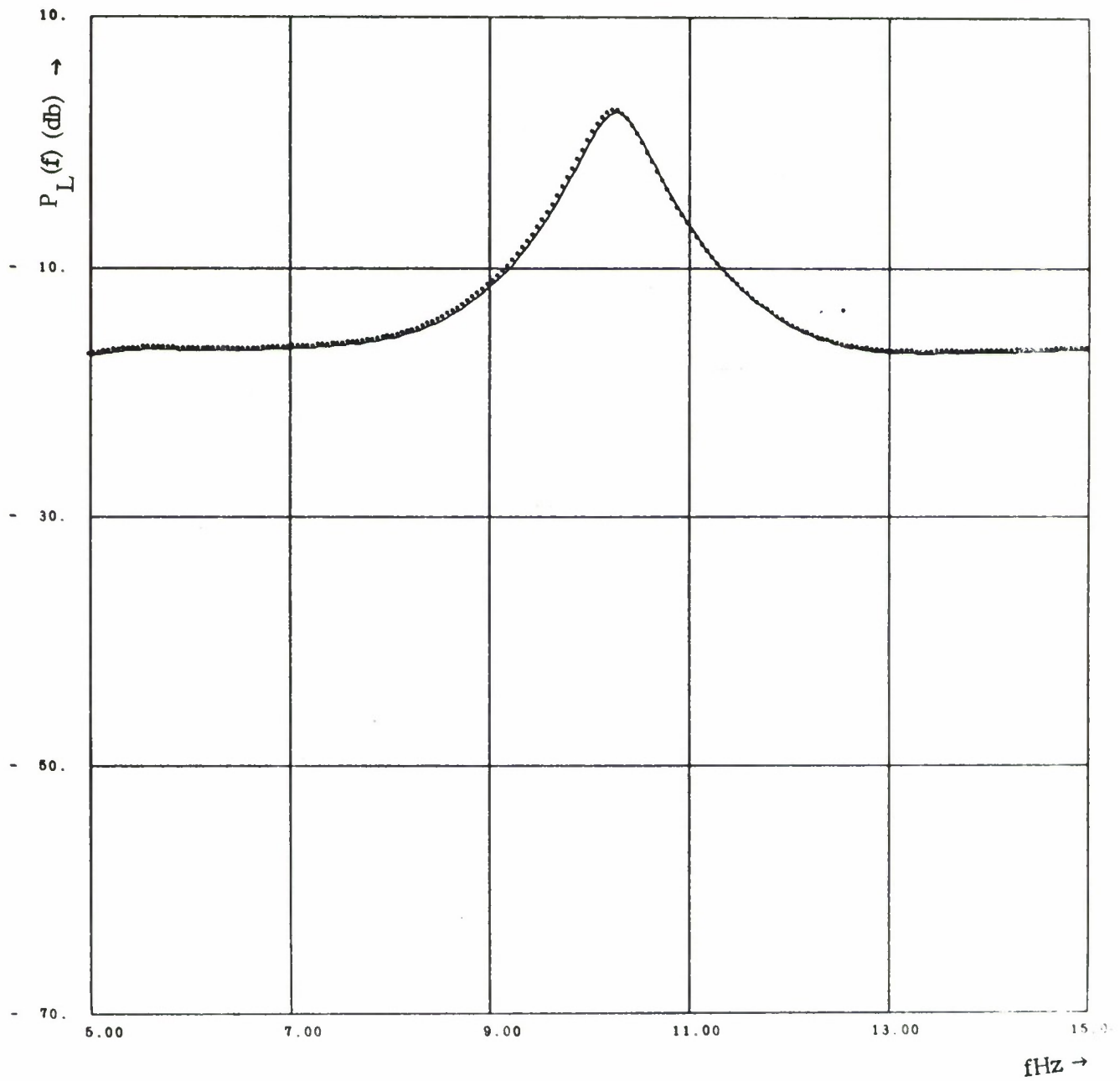
Fig. 5.1 - Indirect, $\rho = .1$.

Fig. 5.2 - Direct, ρ = .1.
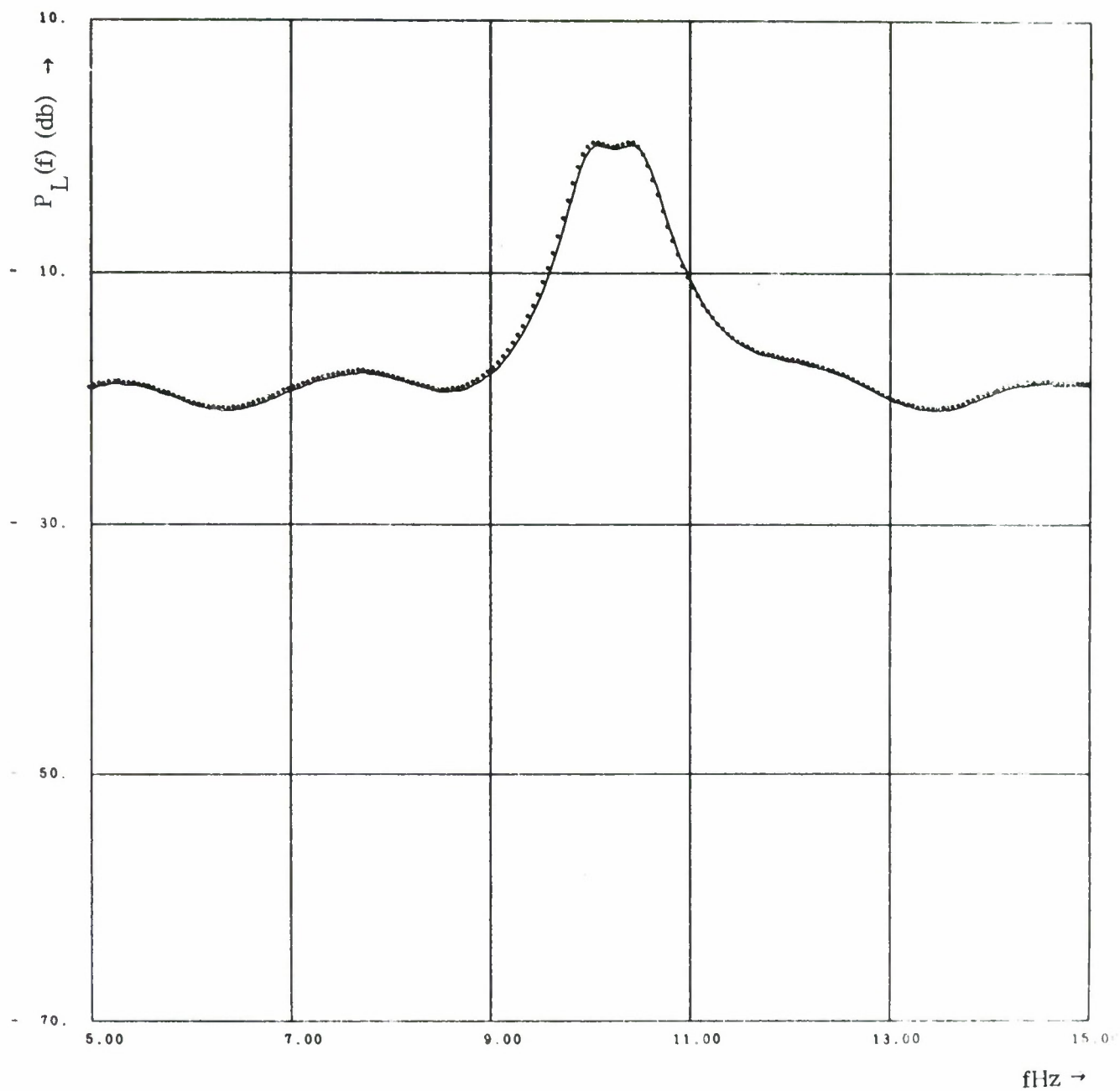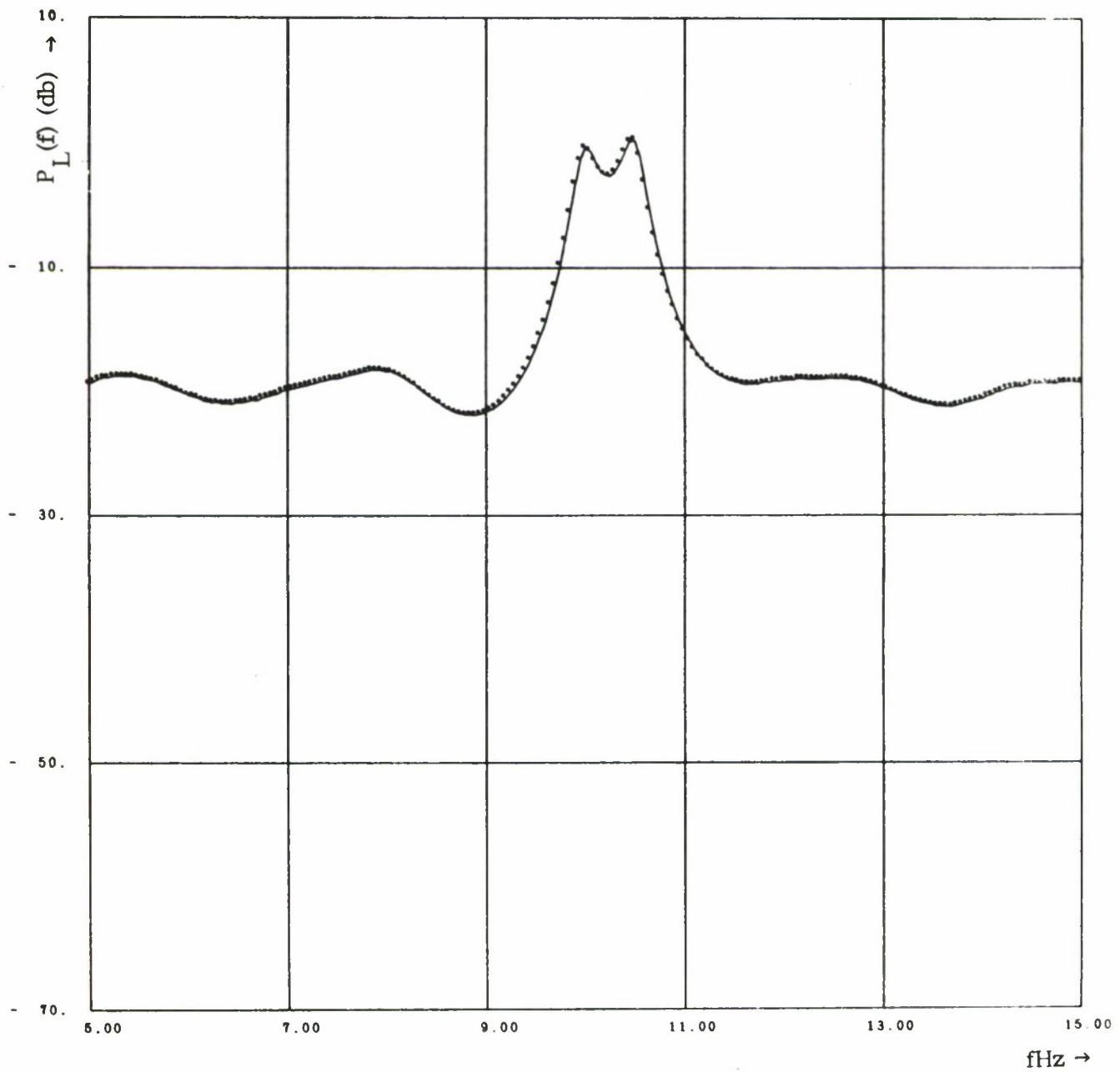
Fig. 5.3 - Indirect, $\rho = .2$.

Fig. 5.4 - Direct, ρ = .2.
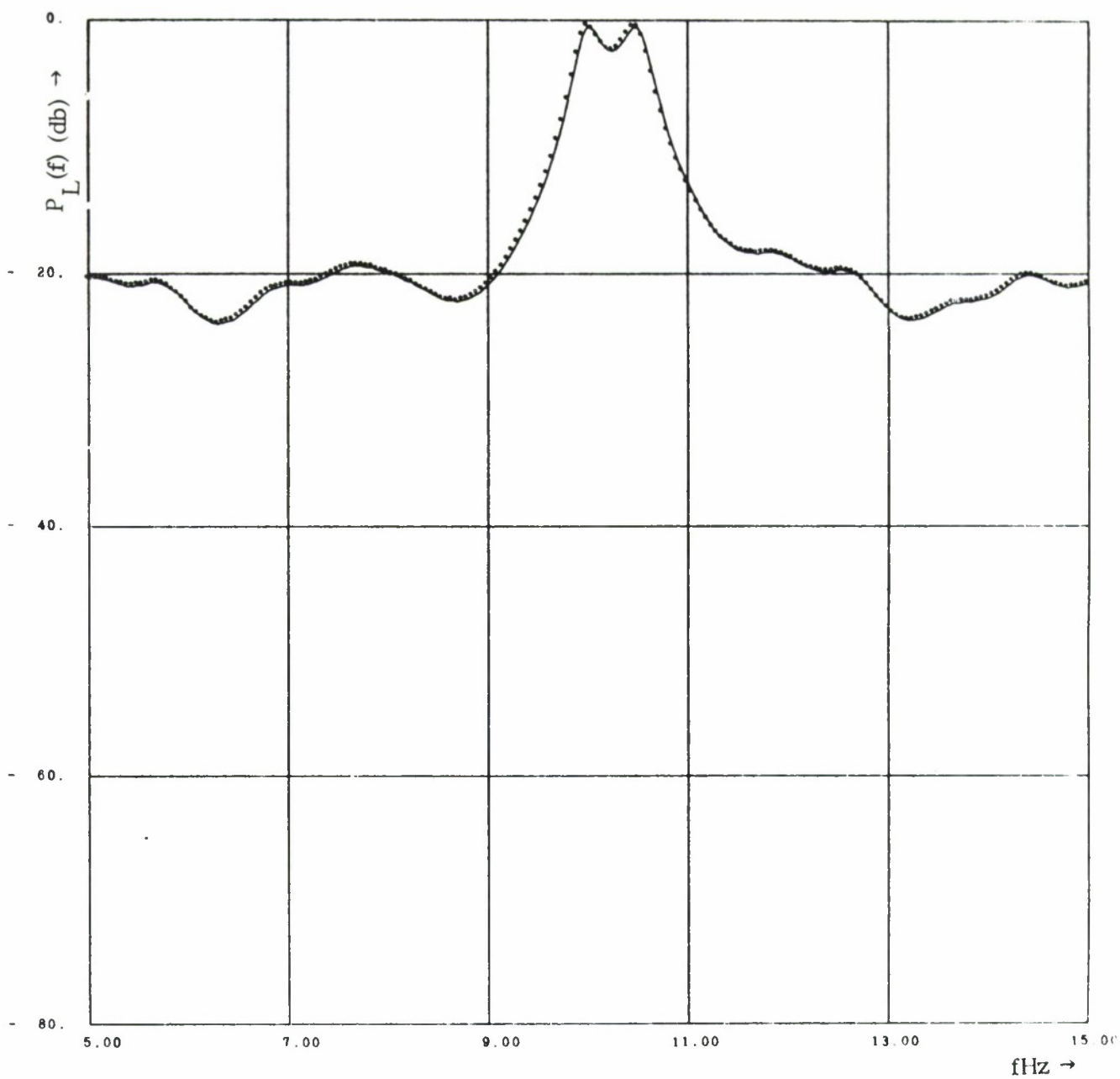
Fig. 5.5 - Indirect, ρ = .3.
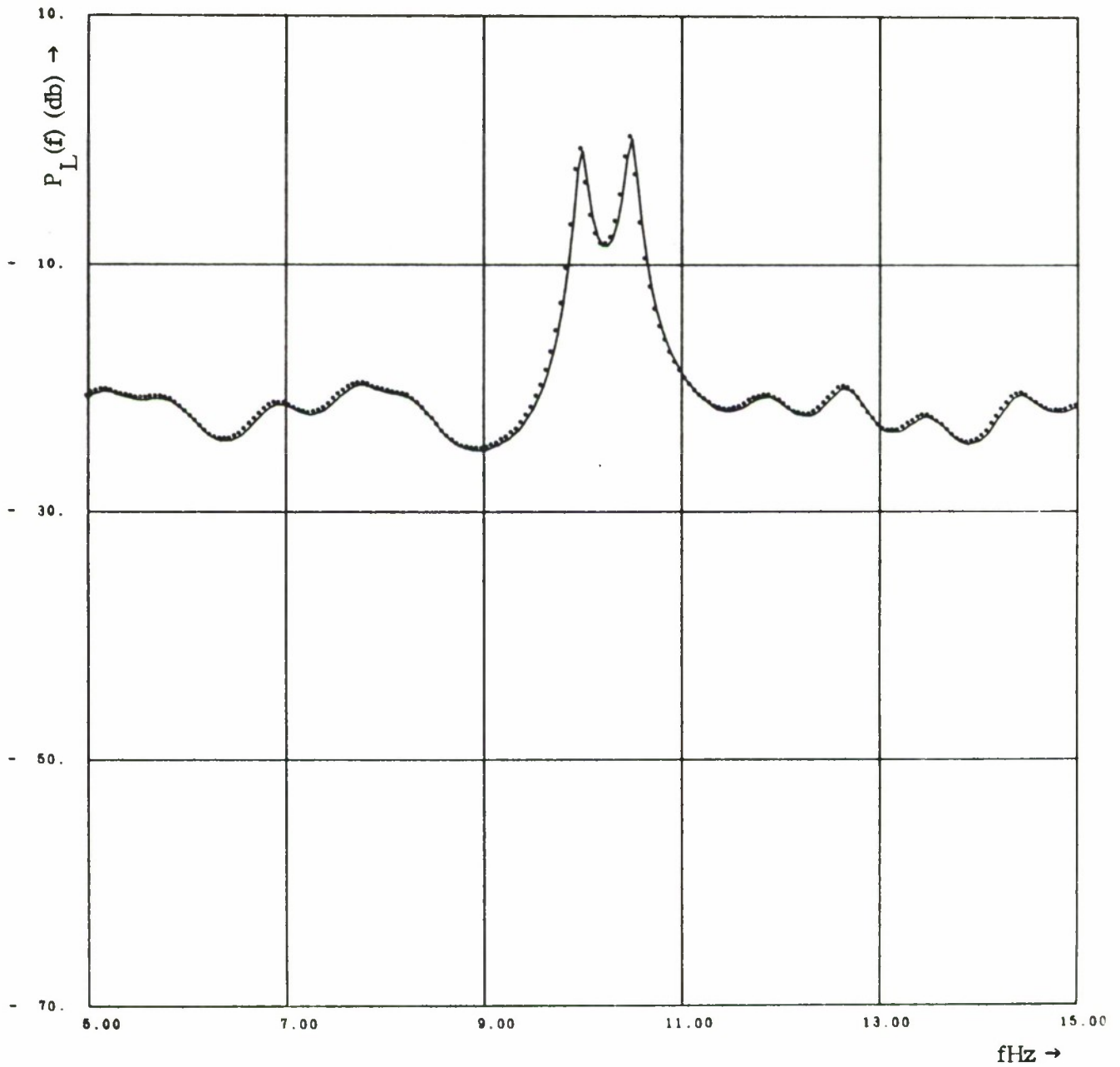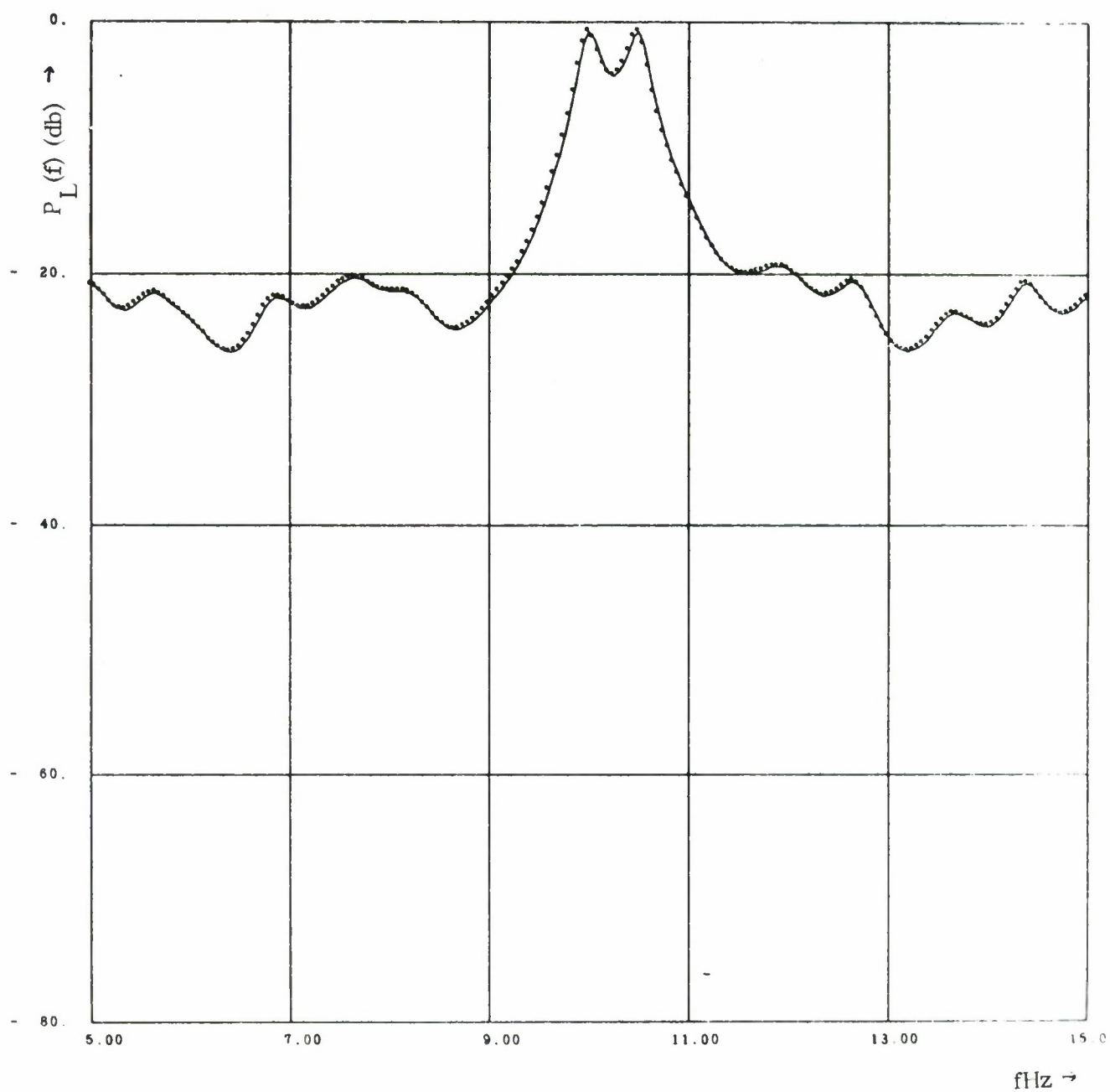
Fig. 5.6 - Direct, $\rho = .3$.

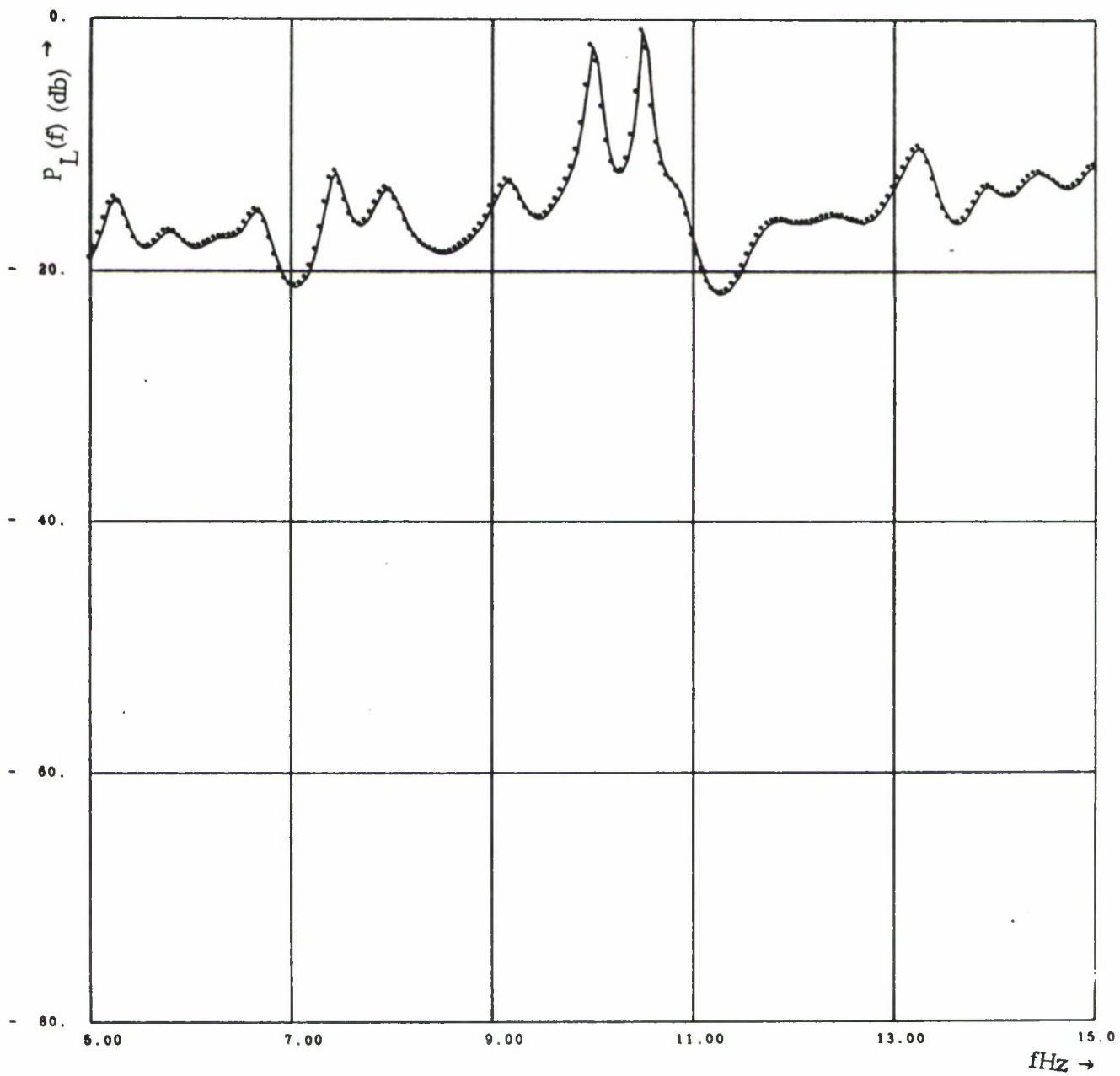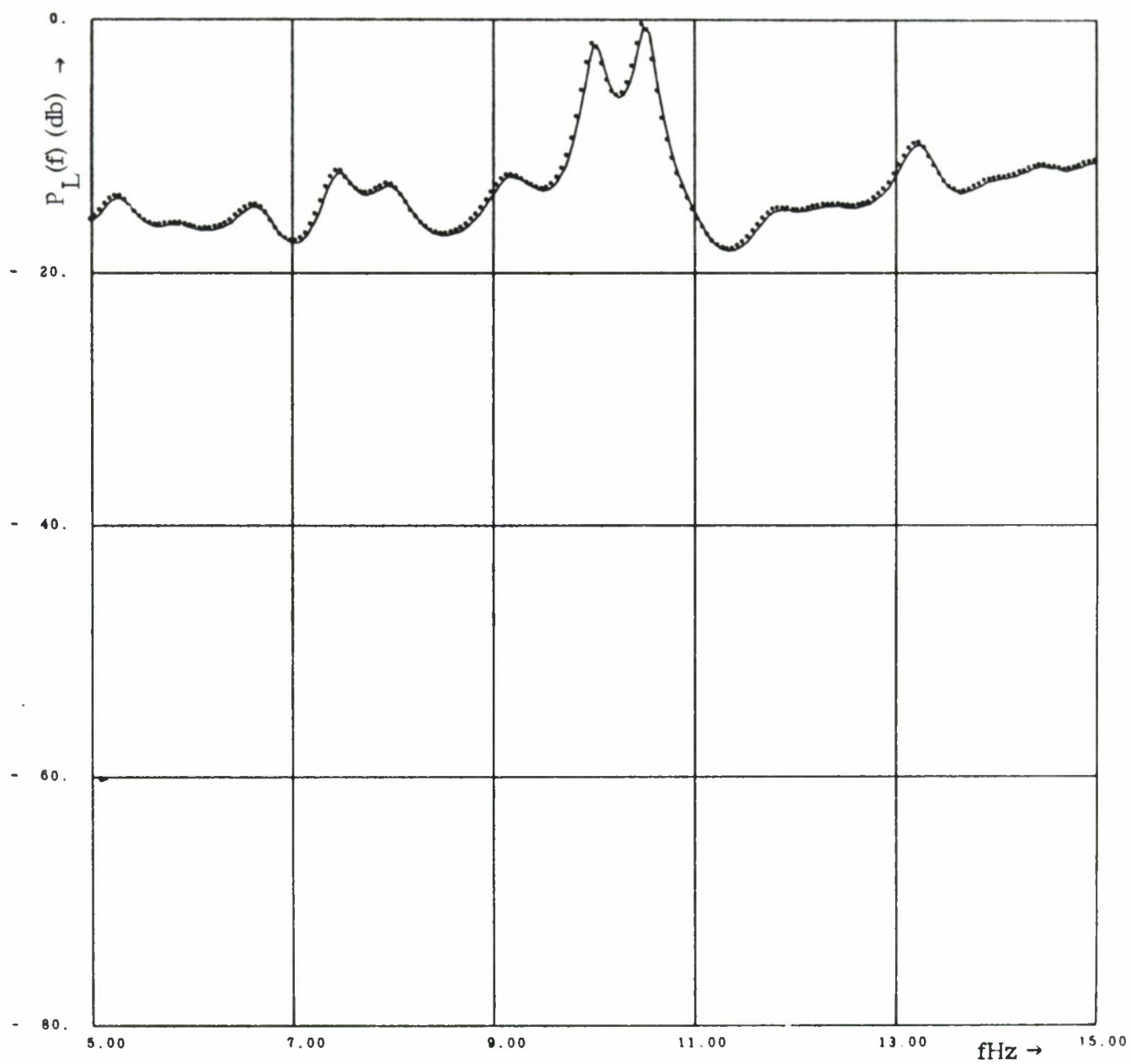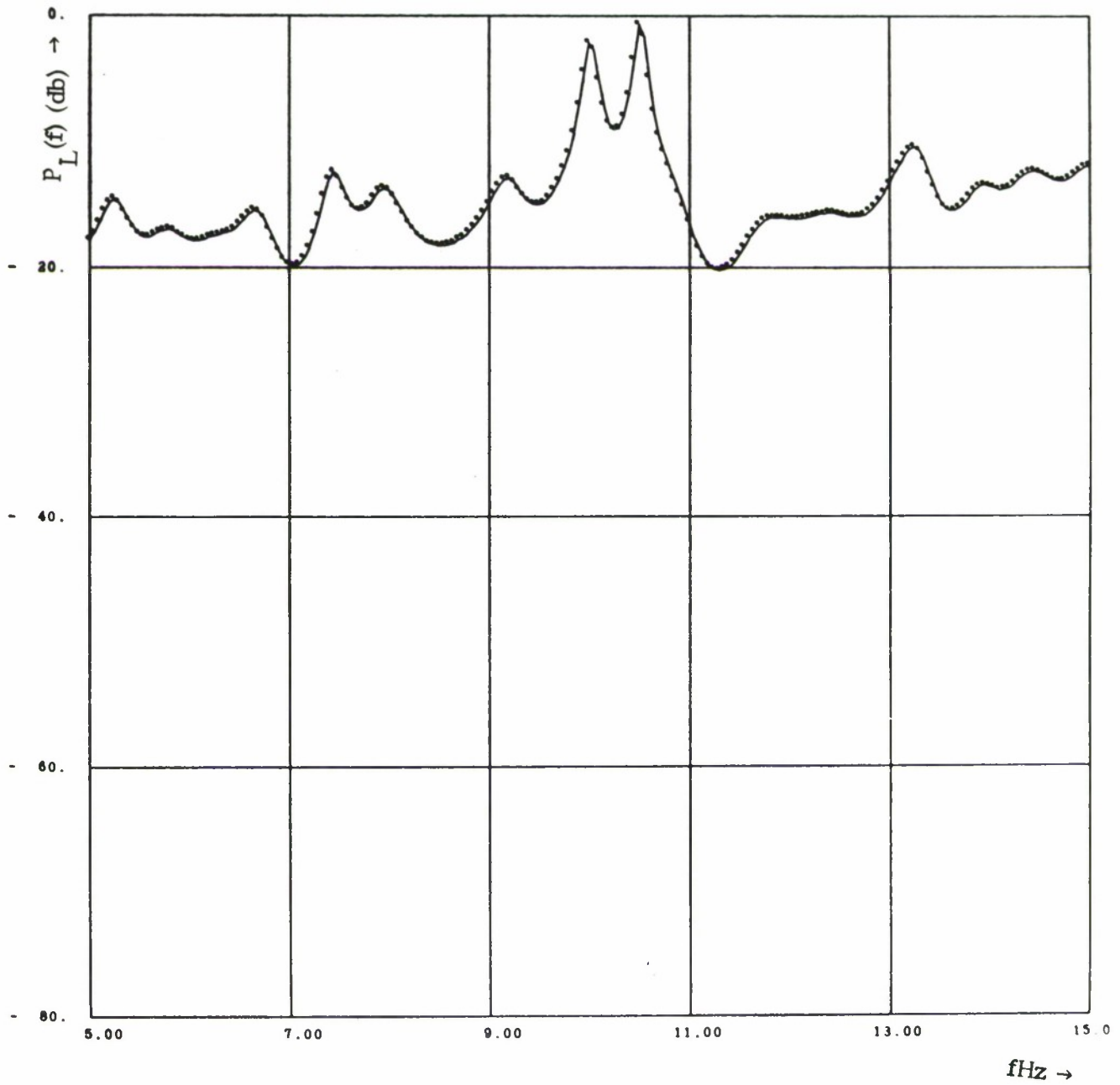Fig. 5.7 - Indirect, $\rho = .4$.

Fig. 5.8 - Direct, $\rho = .4$.

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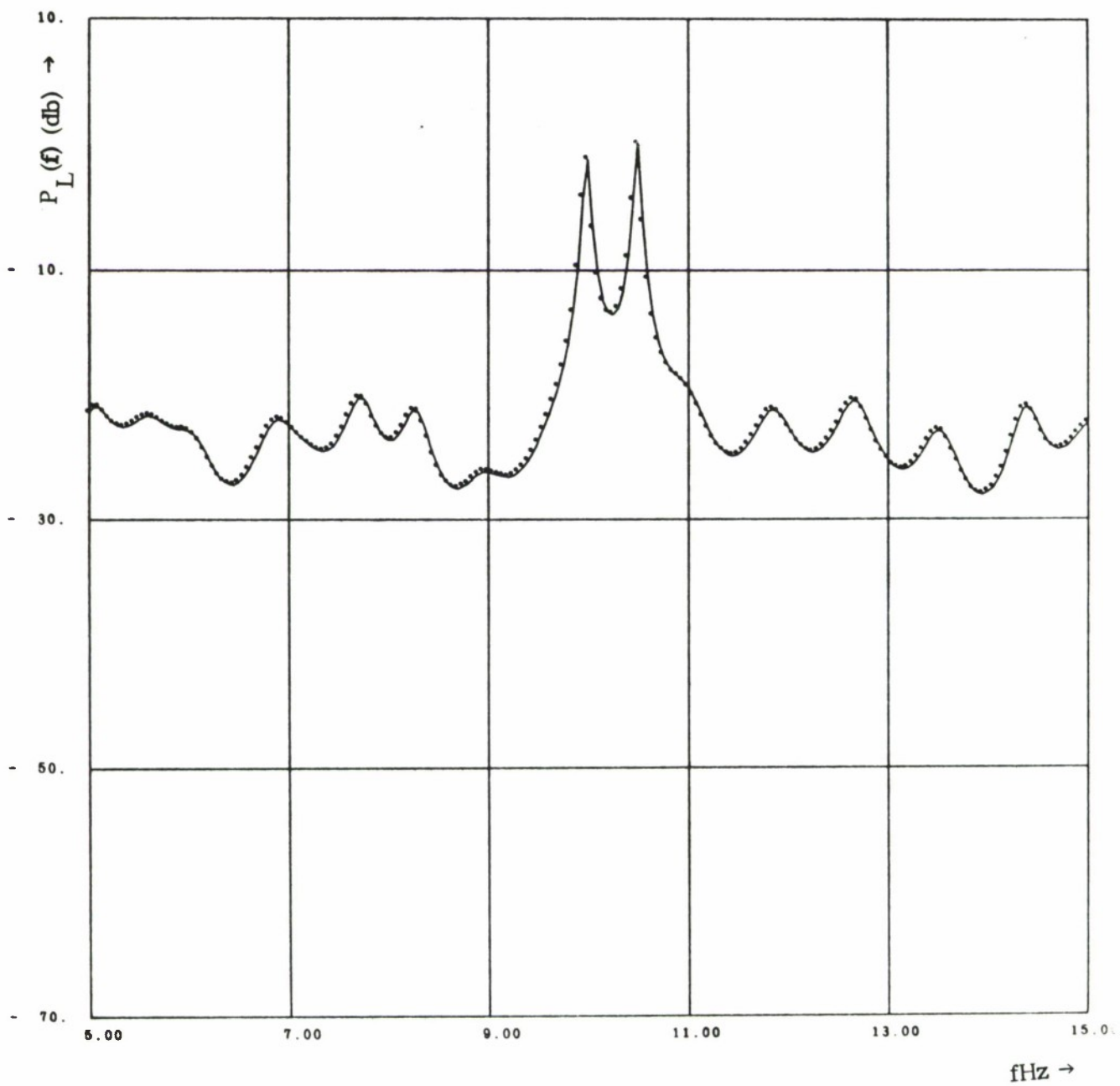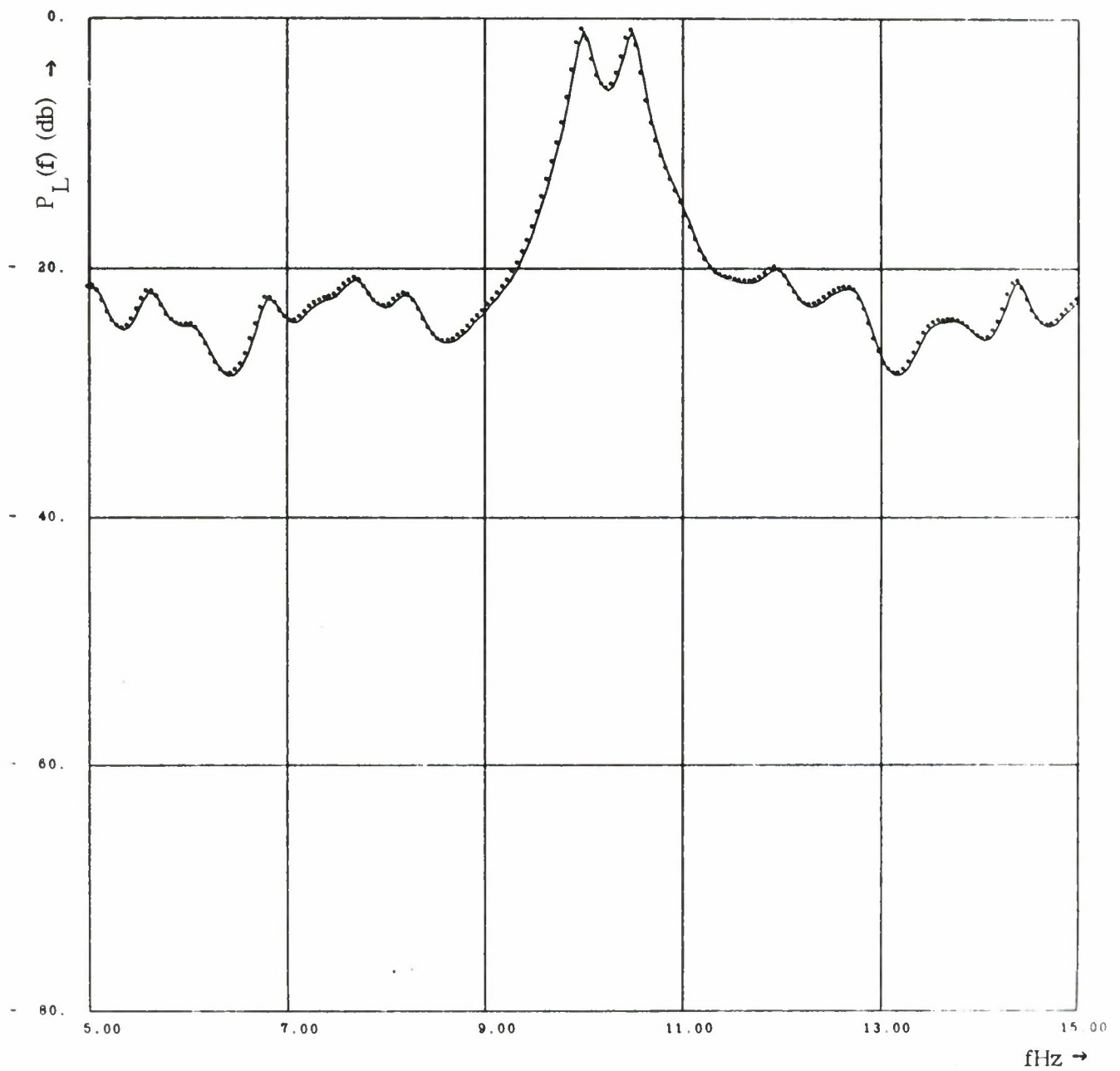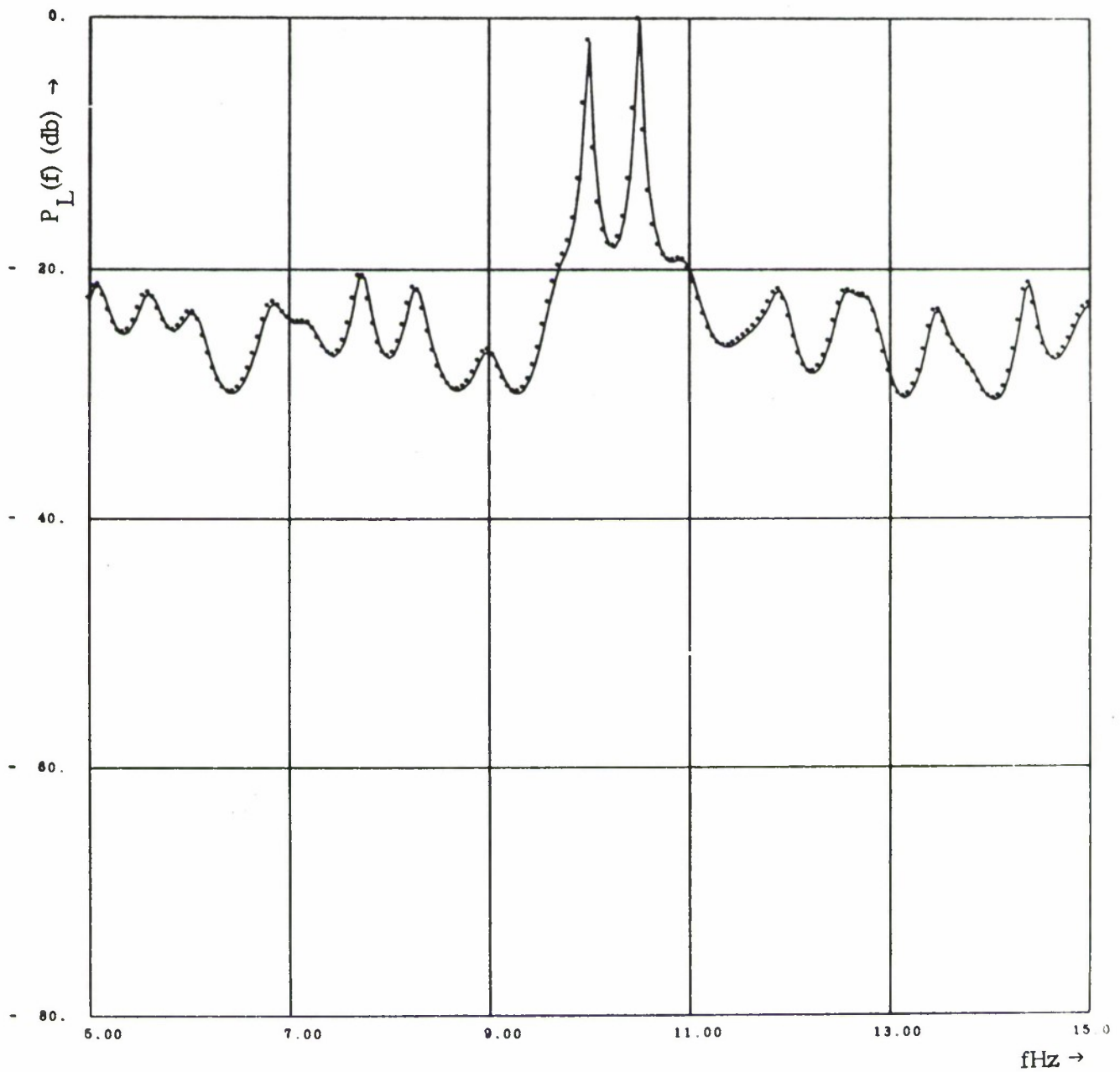
Fig. 5.9 - Indirect, $\rho = .5$.

Fig. 5.10 - Direct, ρ = .5.
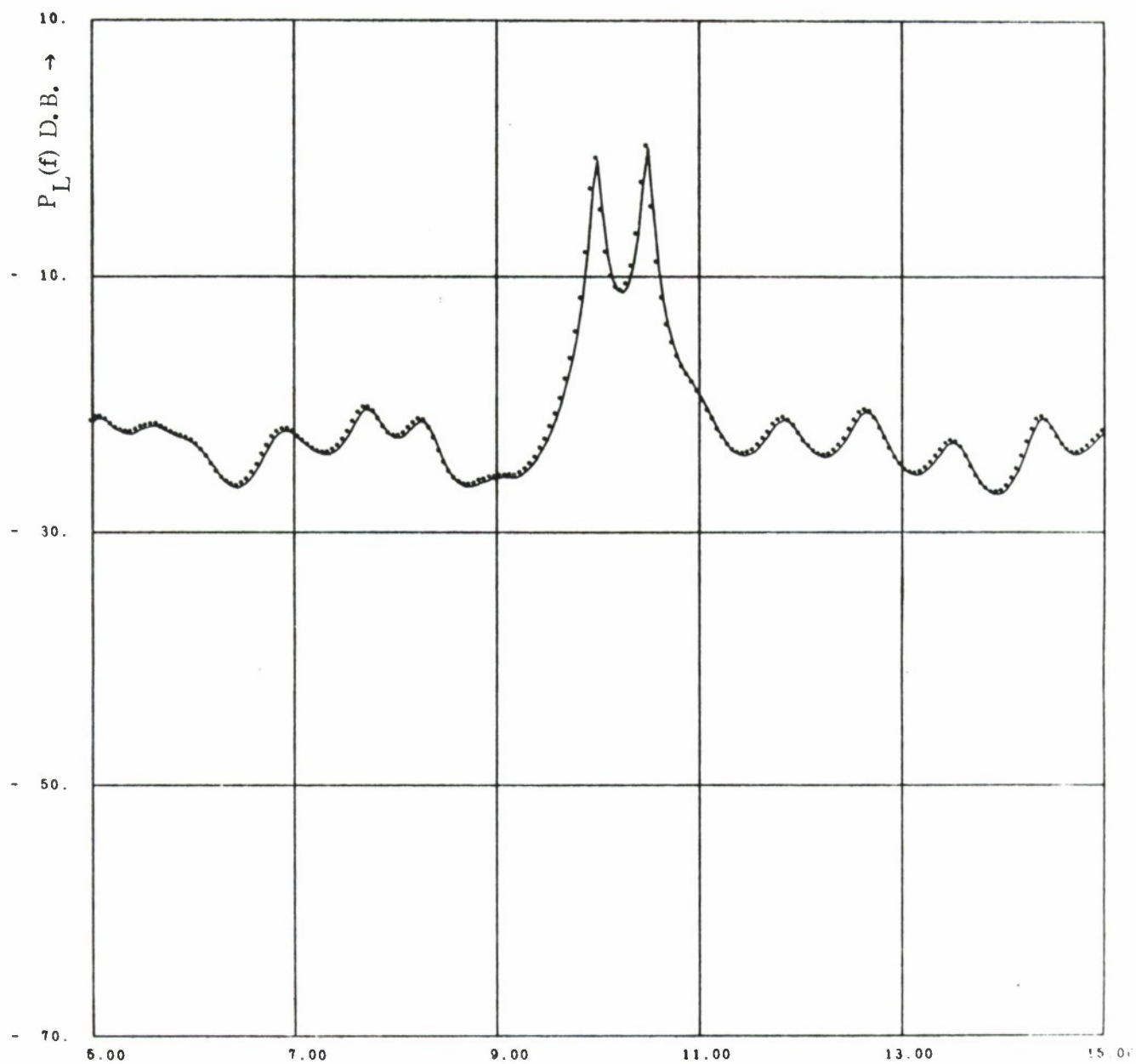
Fig. 5.11 - Direct-Hanning, ρ = .4.
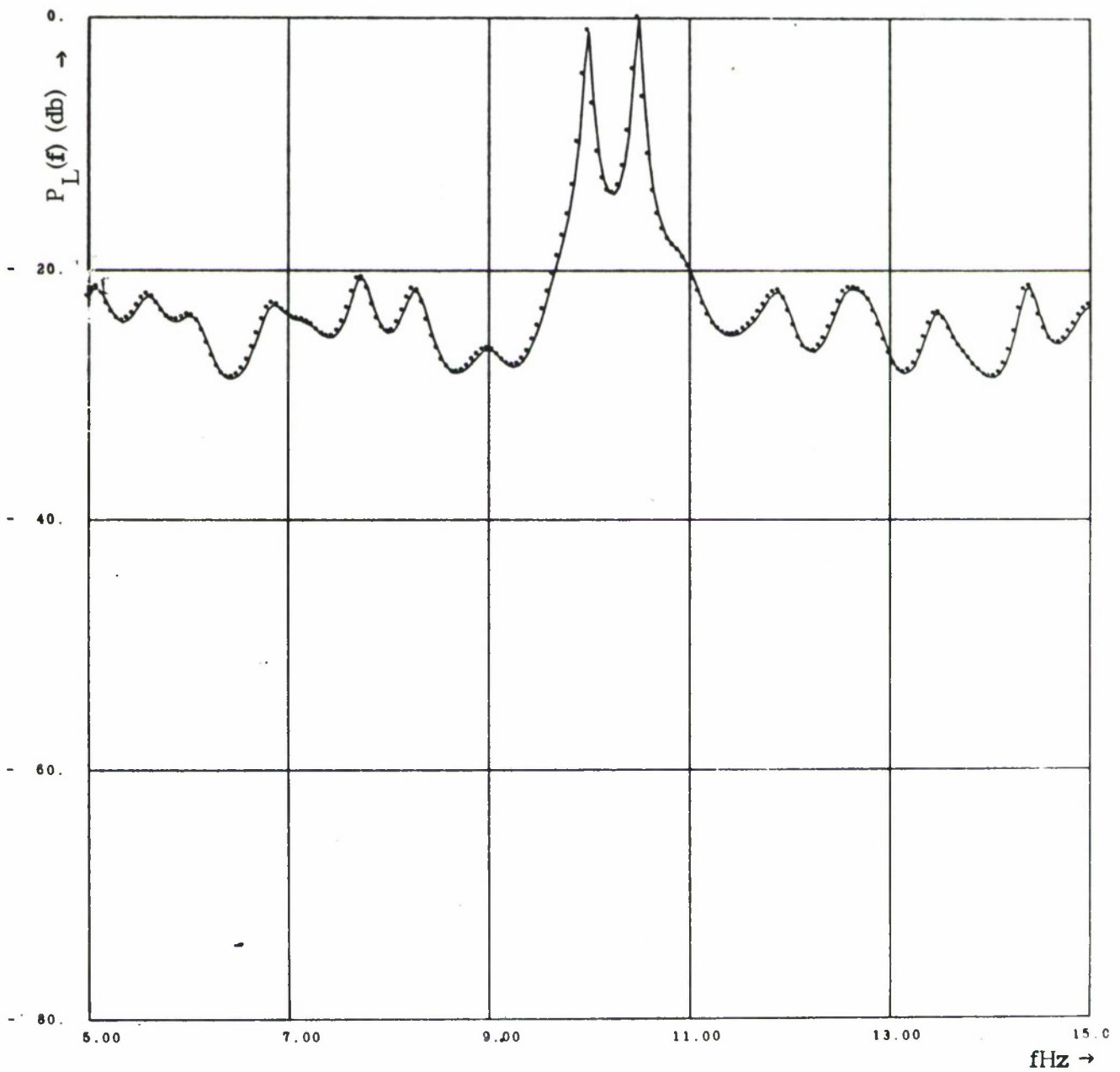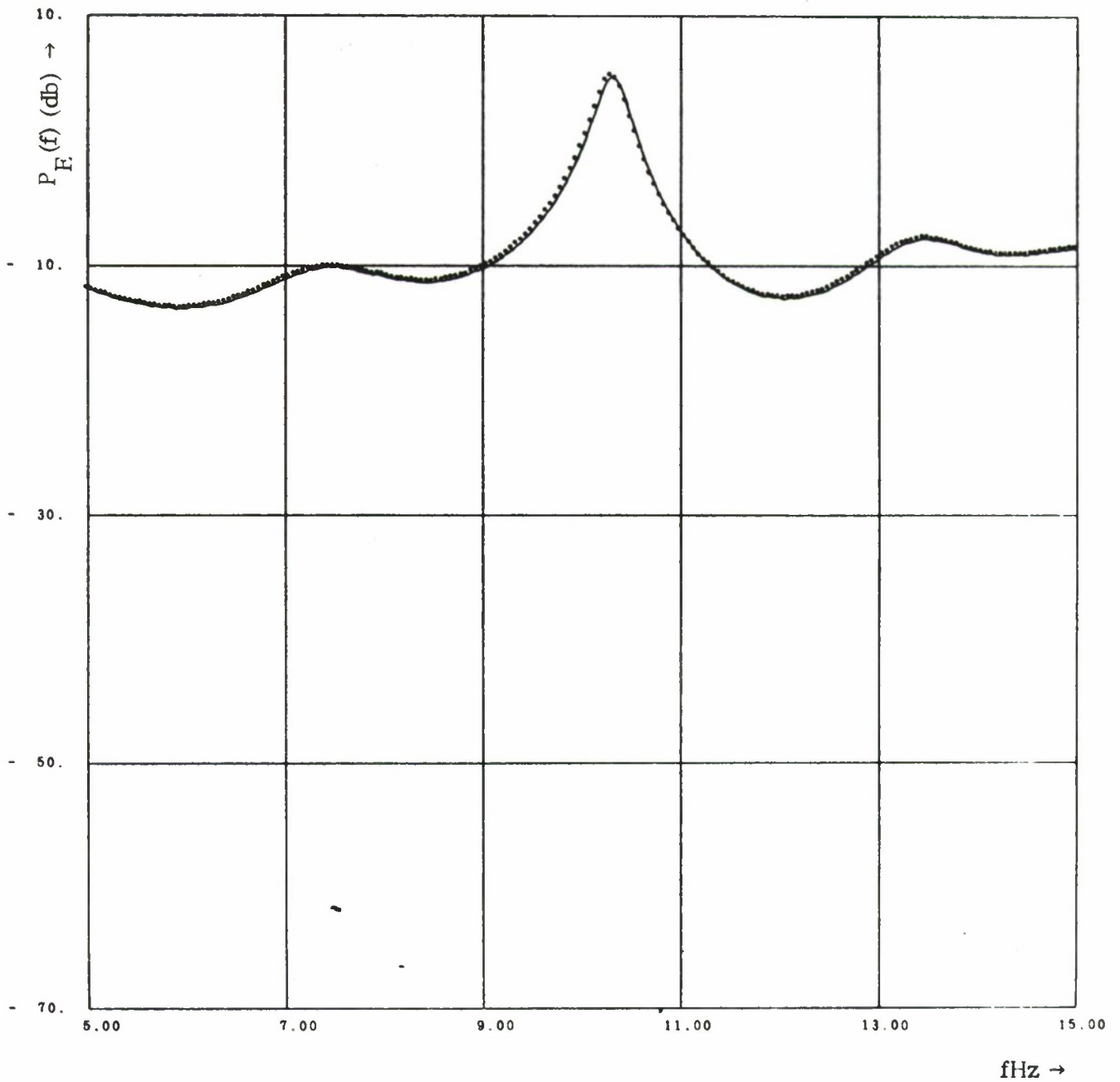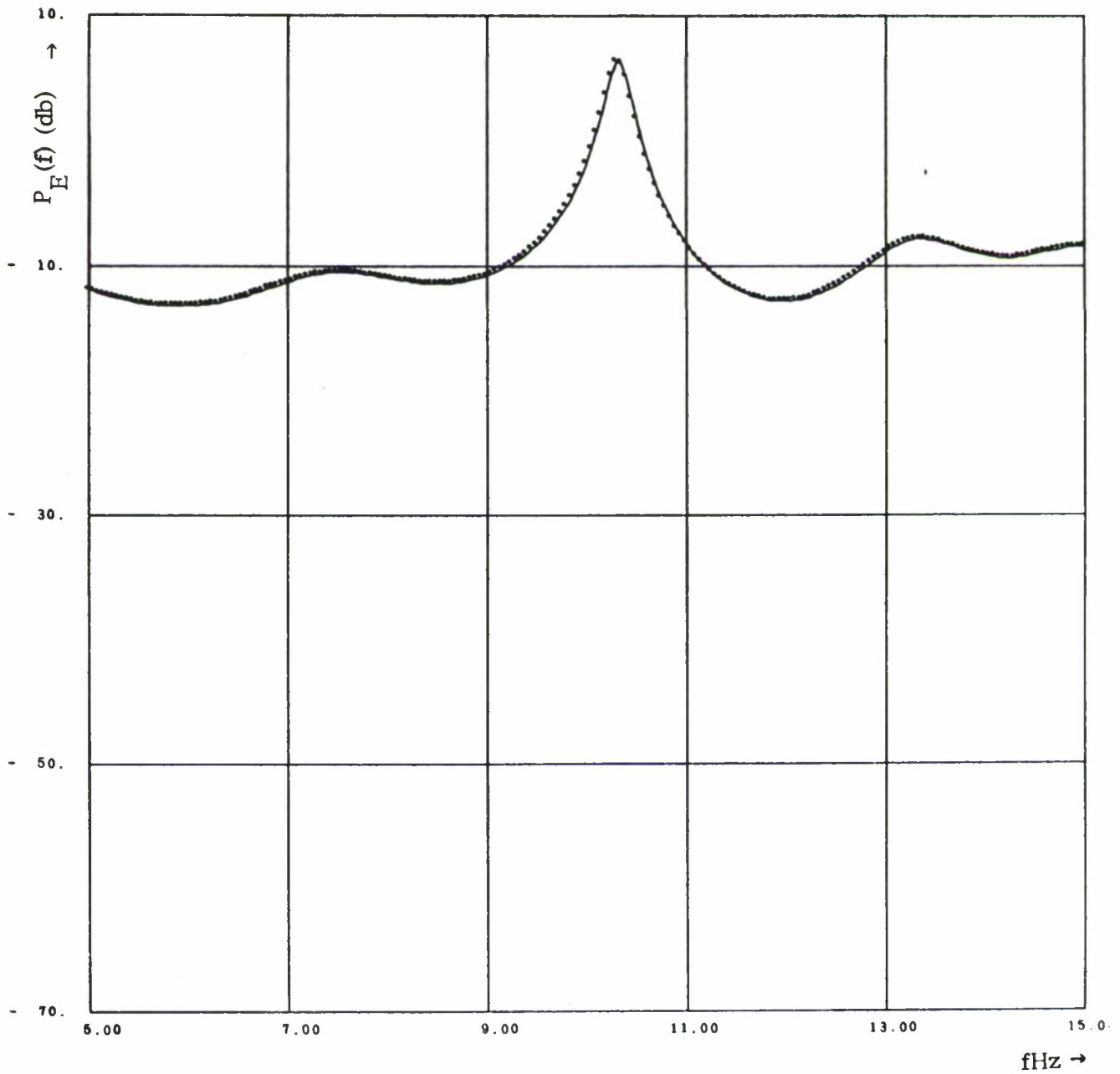
Fig. 5.12 - Direct-Hanning, ρ = .5.
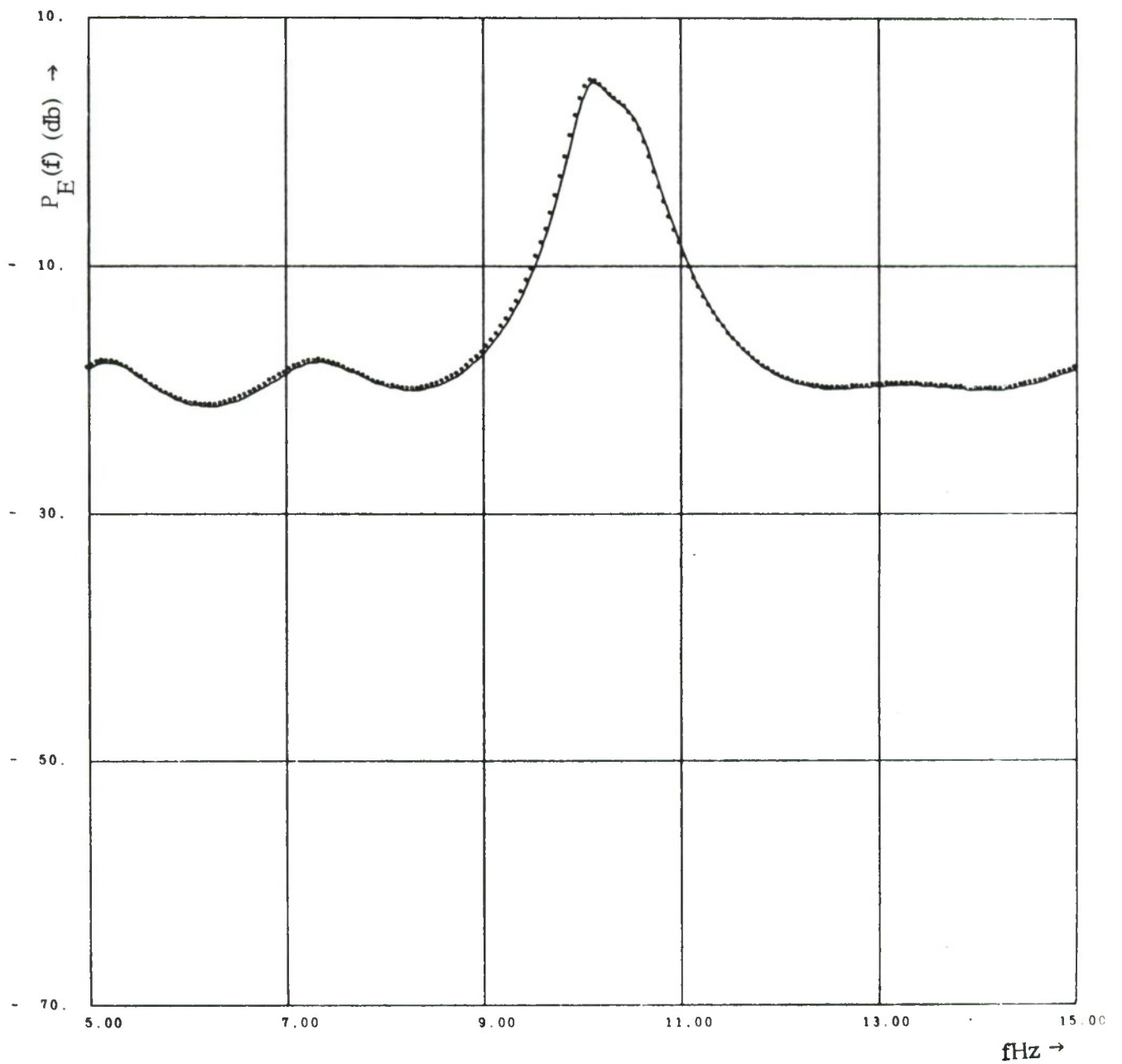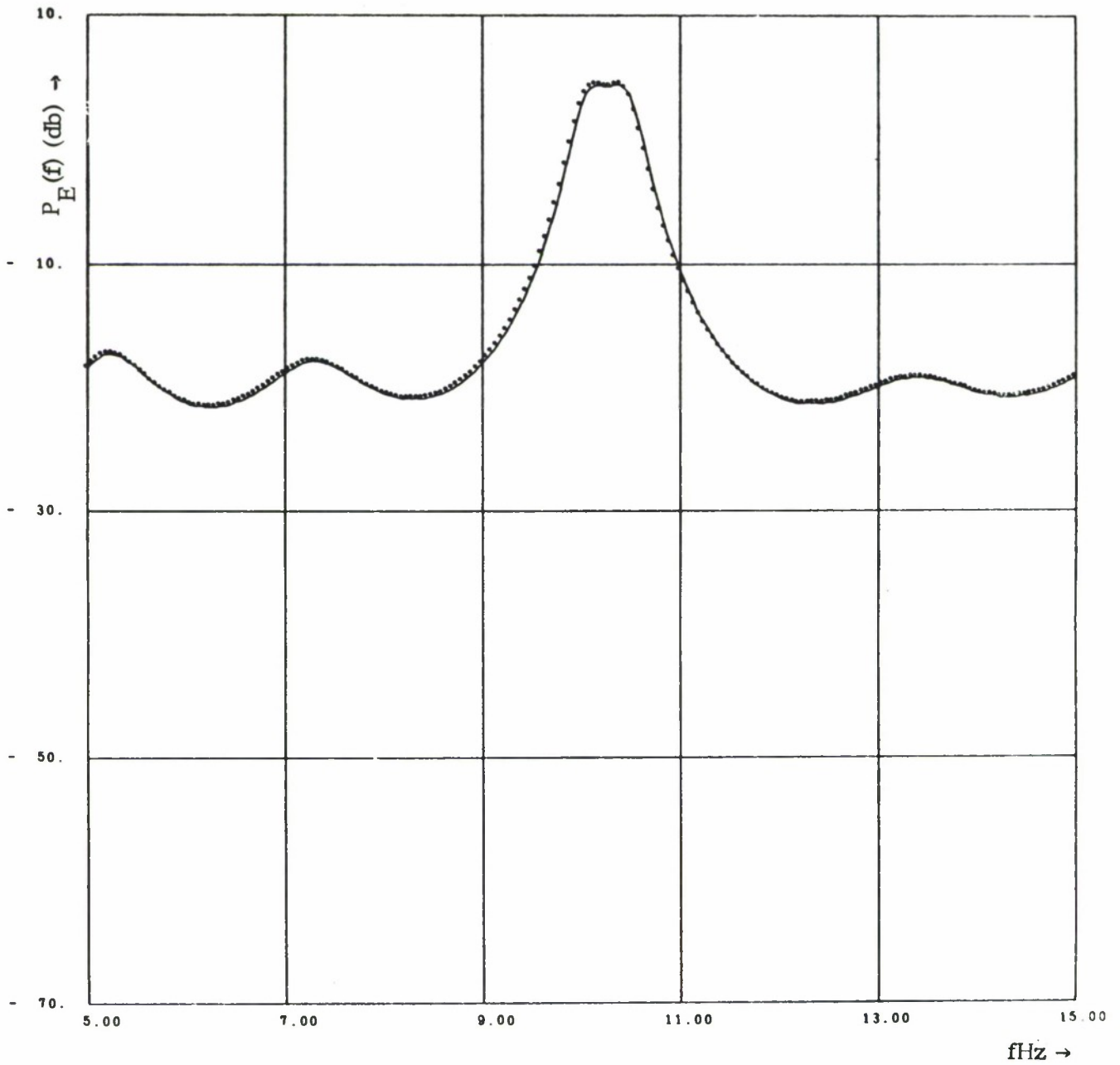
Fig. 5.13 - Indirect, ρ = .1.
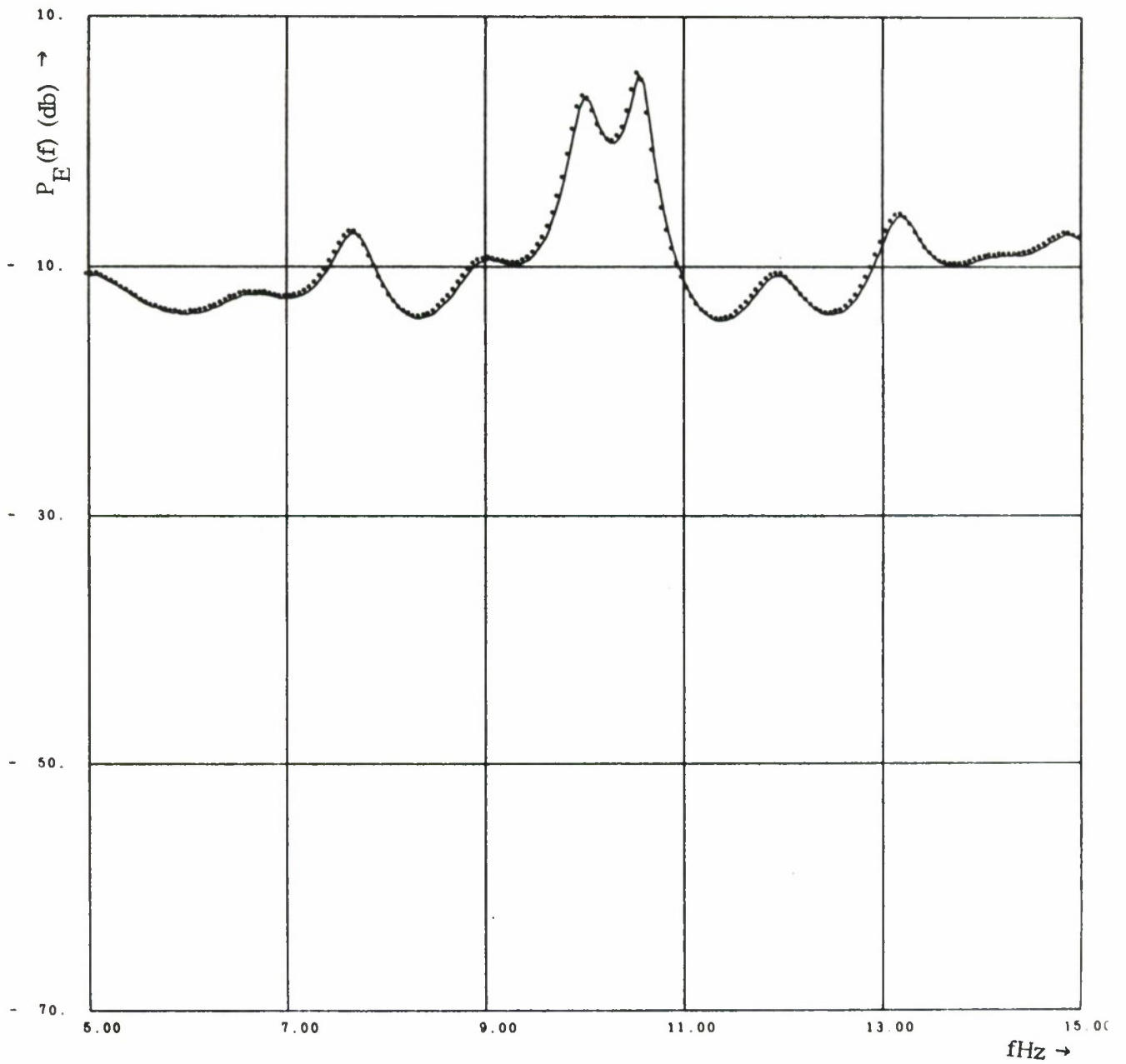
Fig. 5.14 - Direct, ρ = .1.
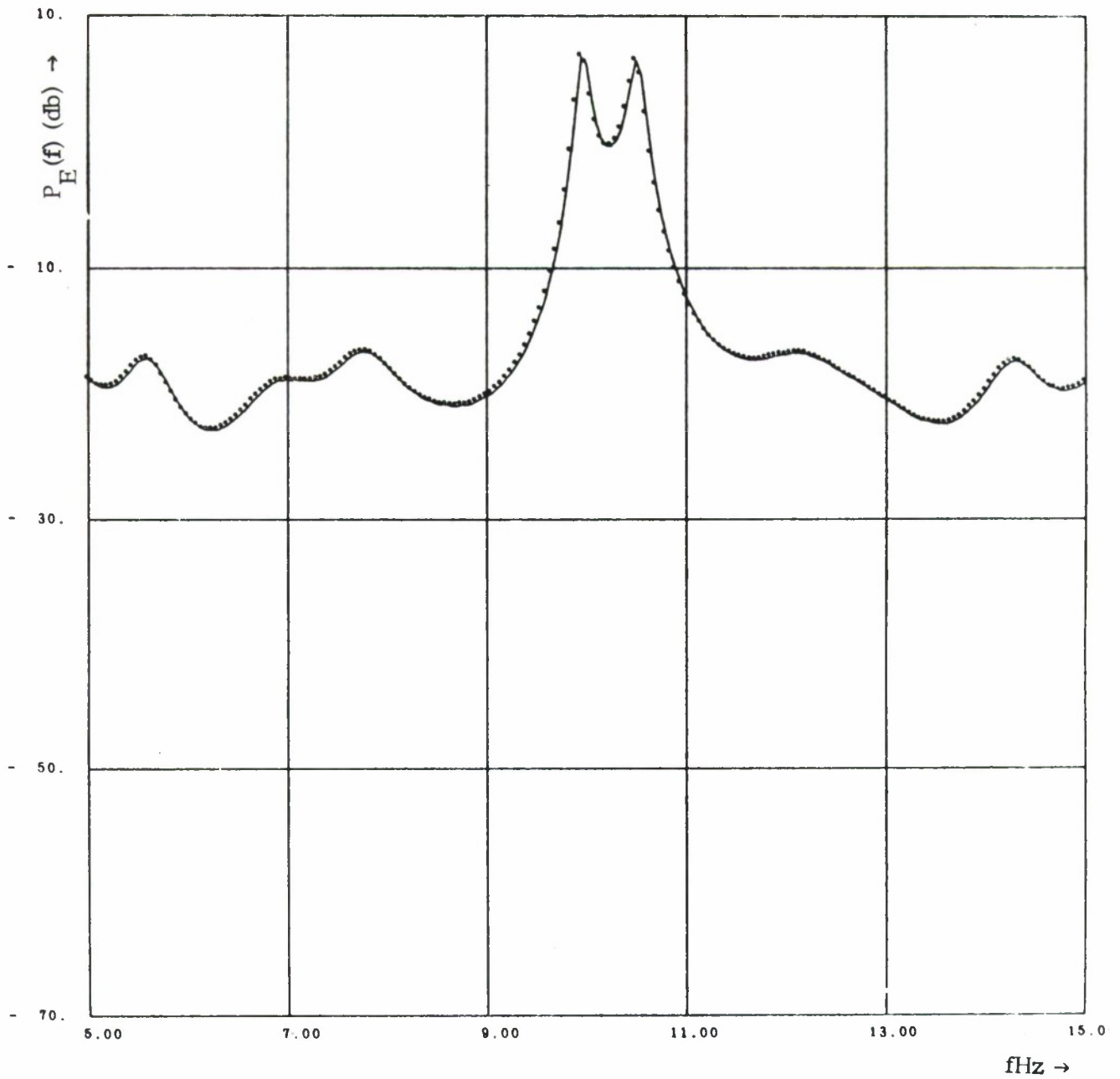
Fig. 5.15 - Indirect, ρ = .2.
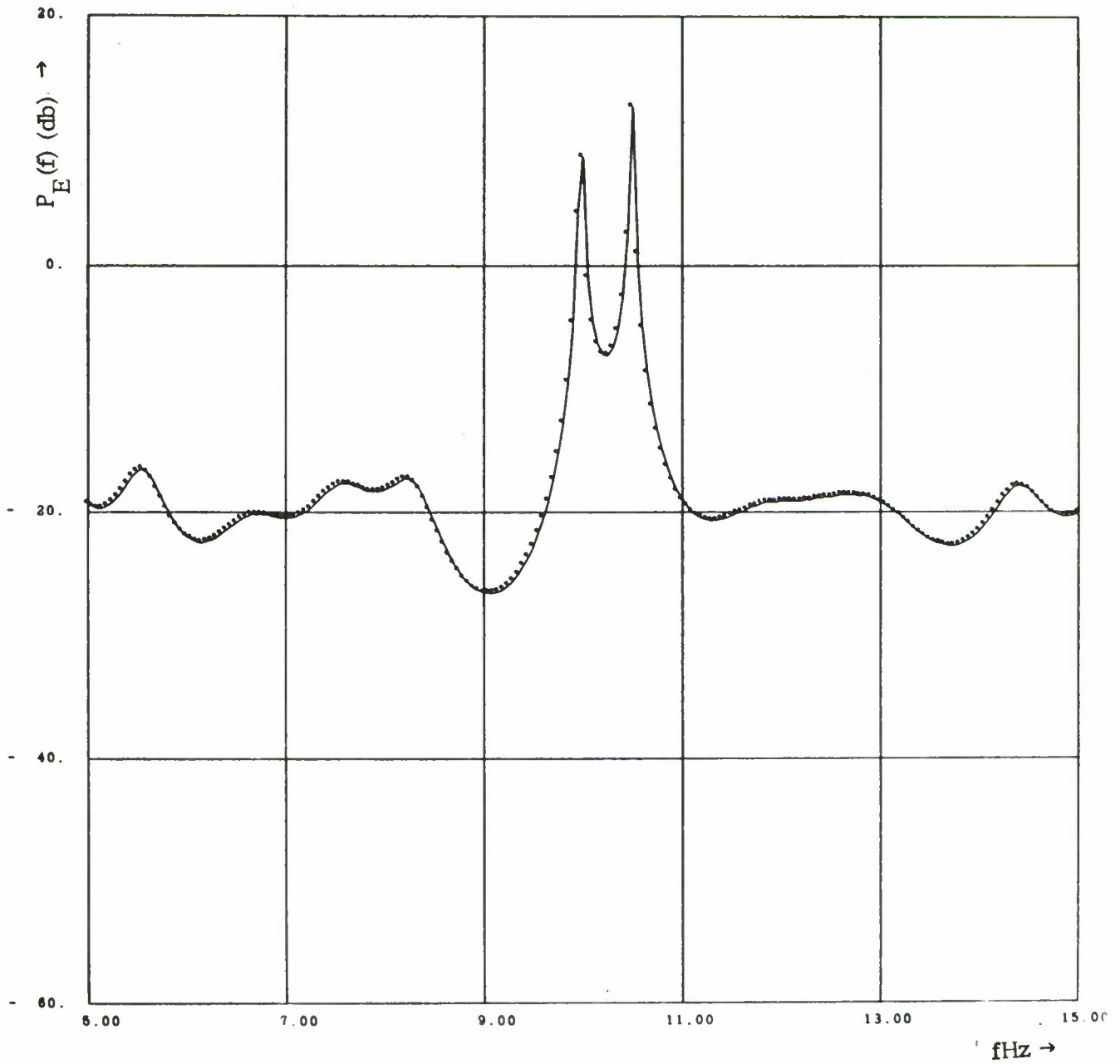
Fig. 5.16 - Direct, ρ = .2.
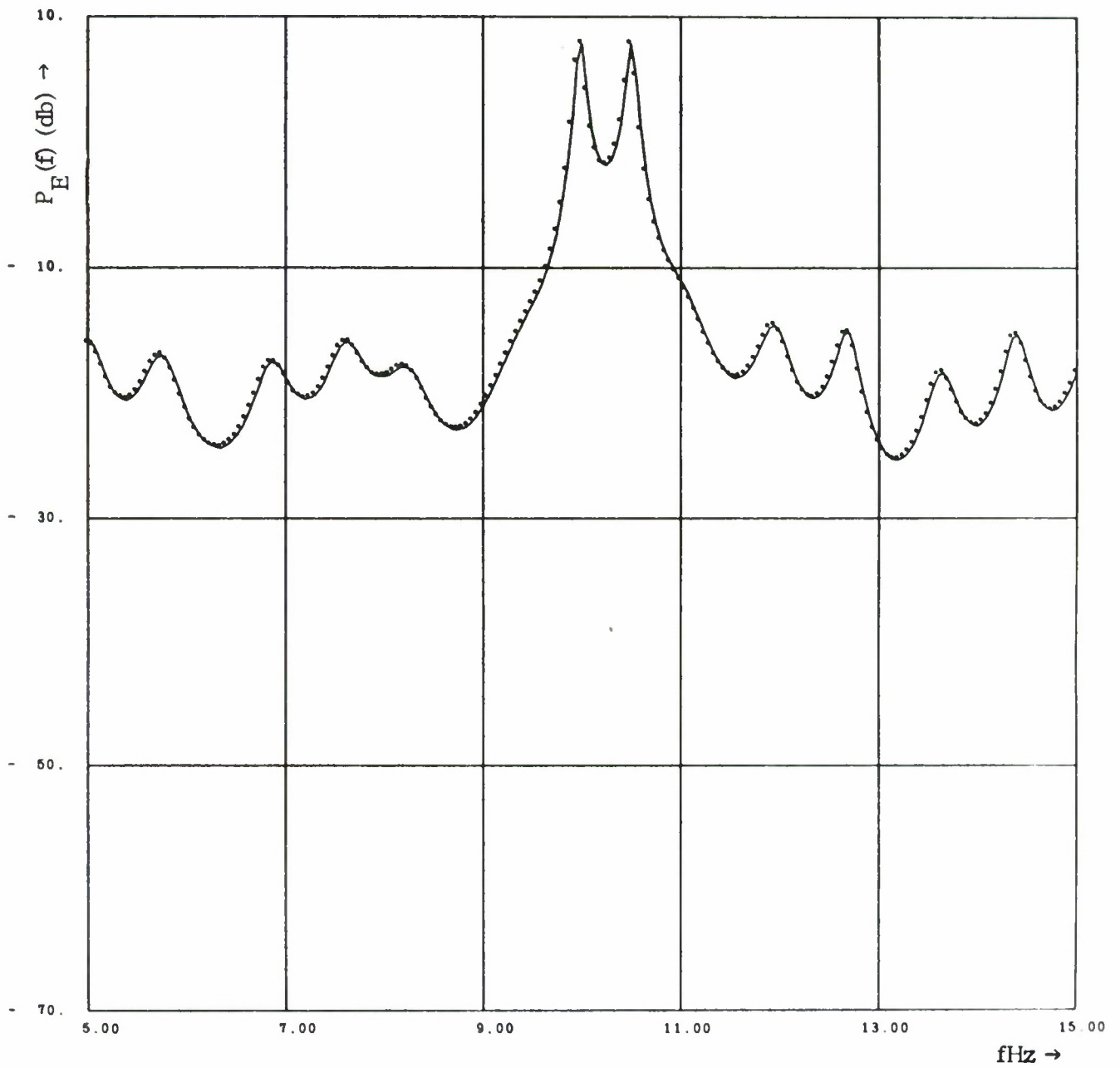
Fig. 5.17 - Indirect, ρ = .3.
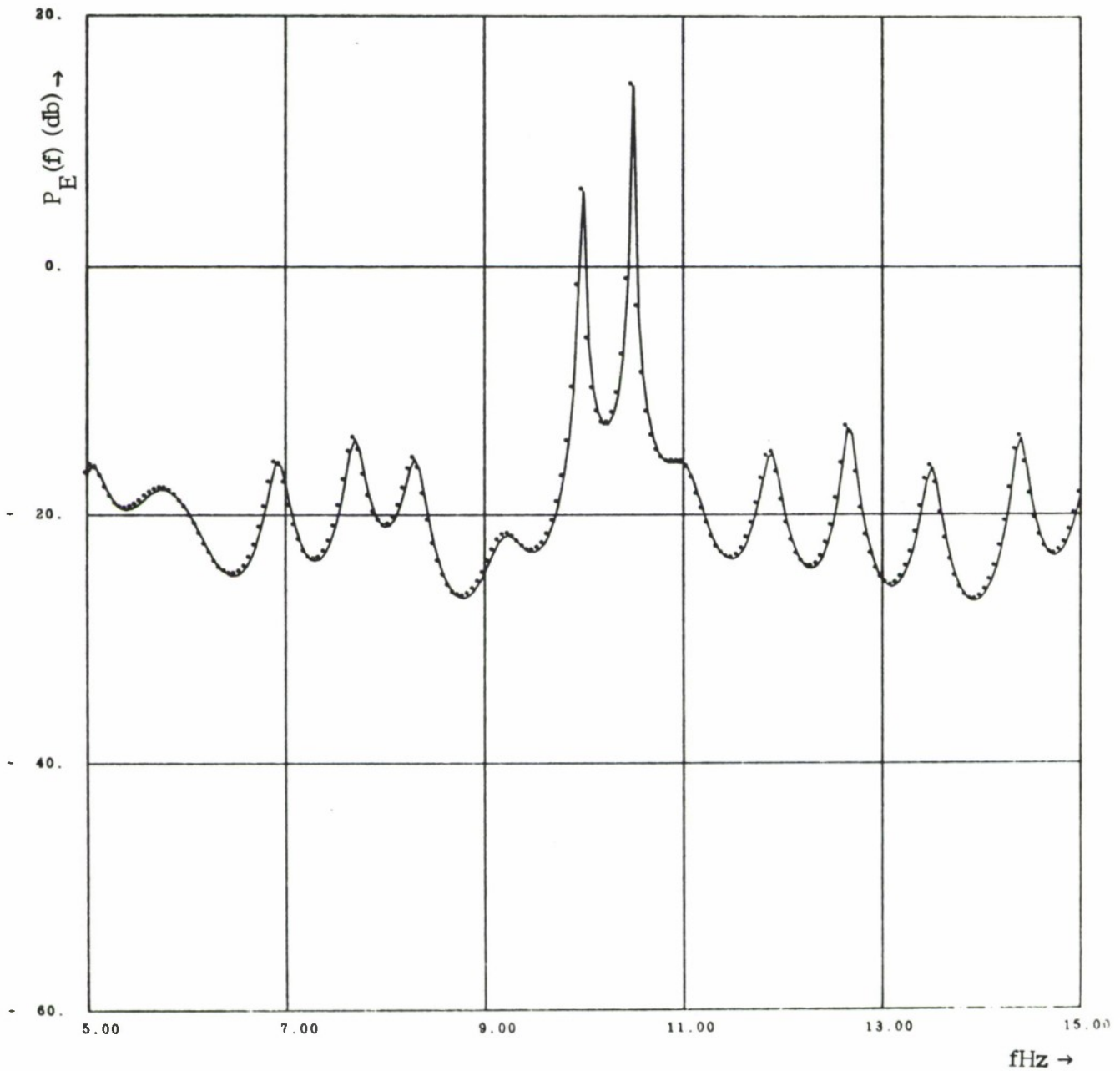
Fig. 5.18 - Direct, ρ = .3.
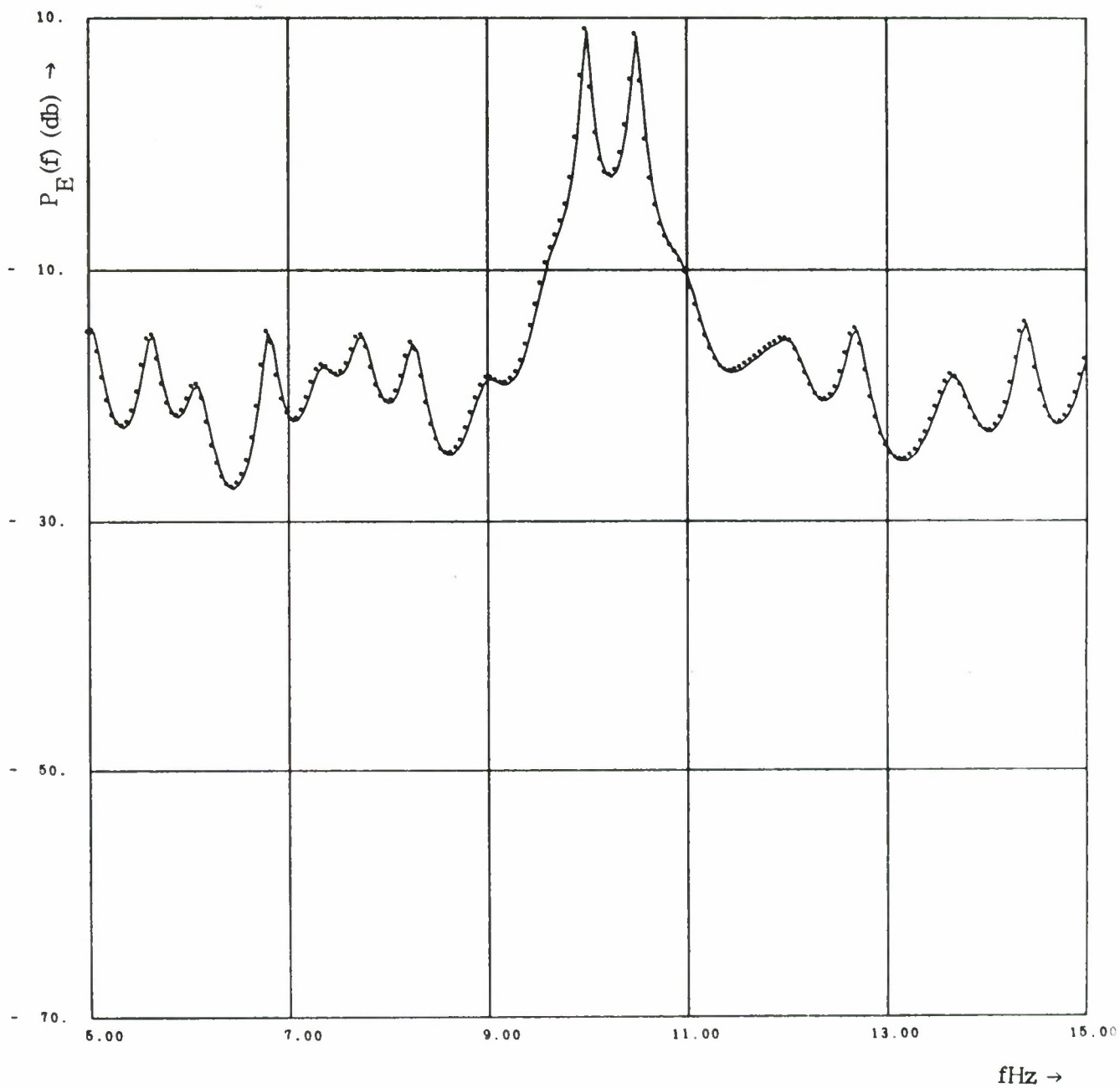
Fig. 5.19 - Indirect, $\rho = .4.$
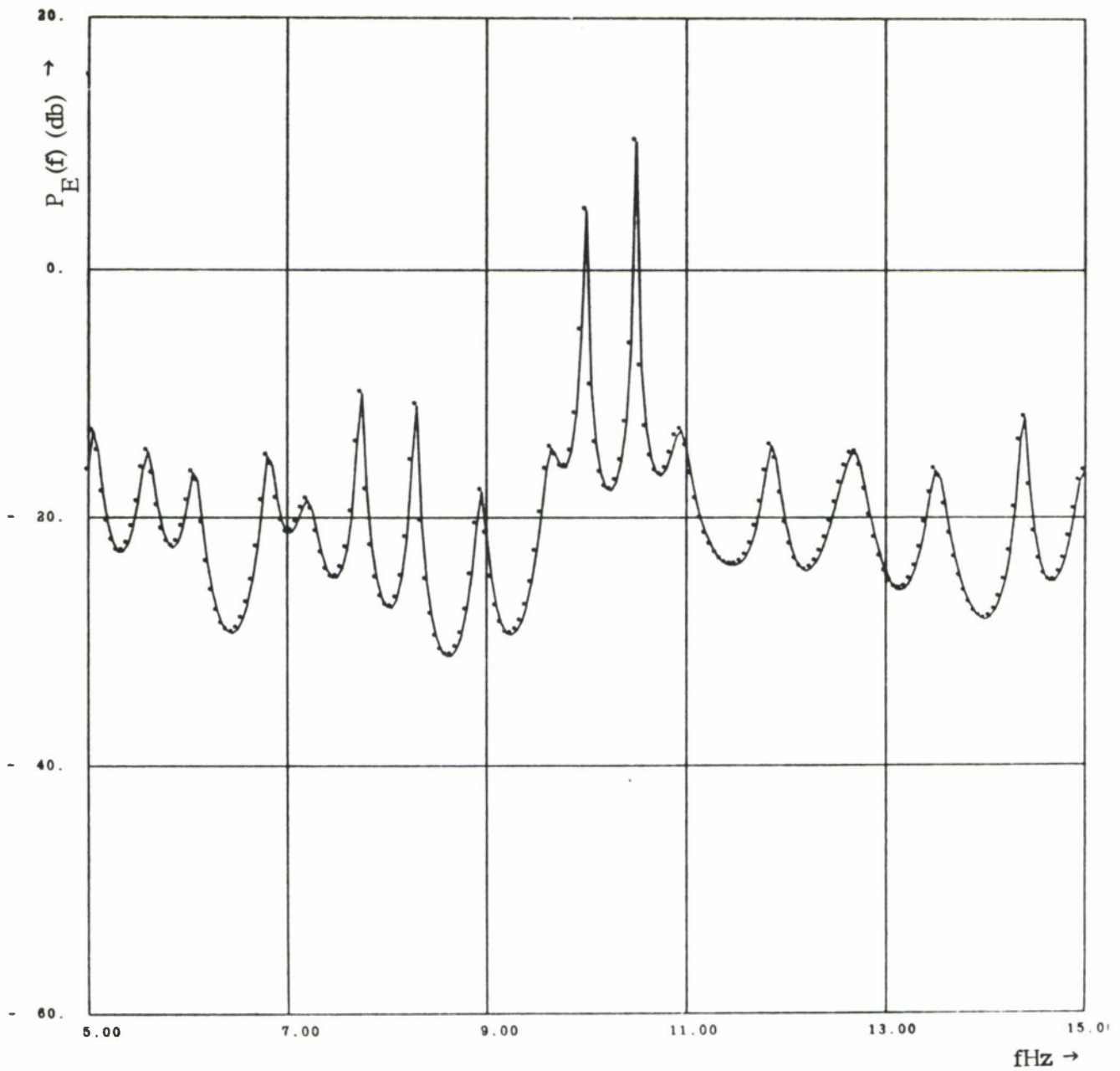
Fig. 5.20 - Direct, ρ = .4.
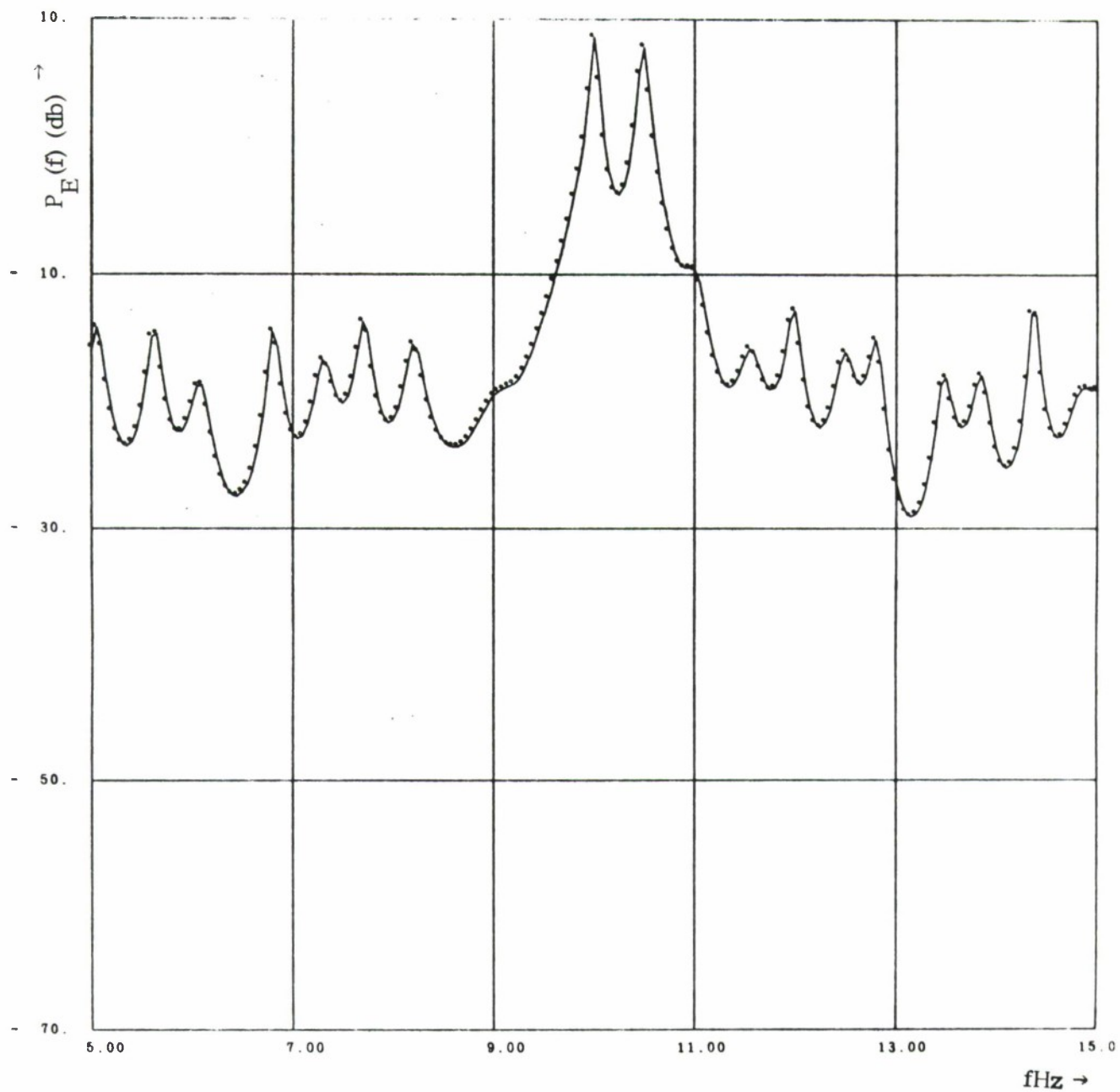
Fig. 5.21 - Indirect, ρ = .5.
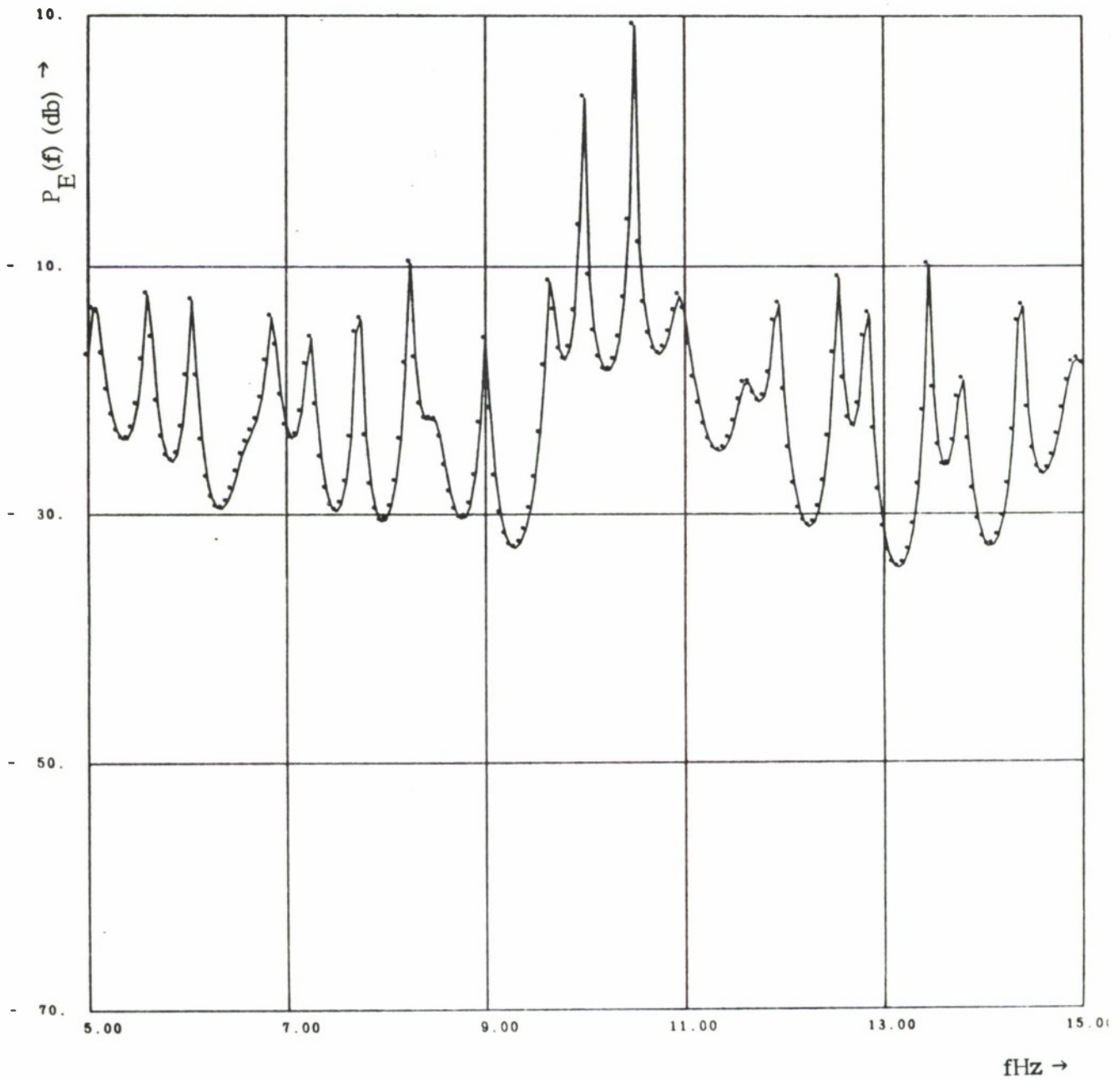
Fig. 5.22 - Direct, ρ = .5.
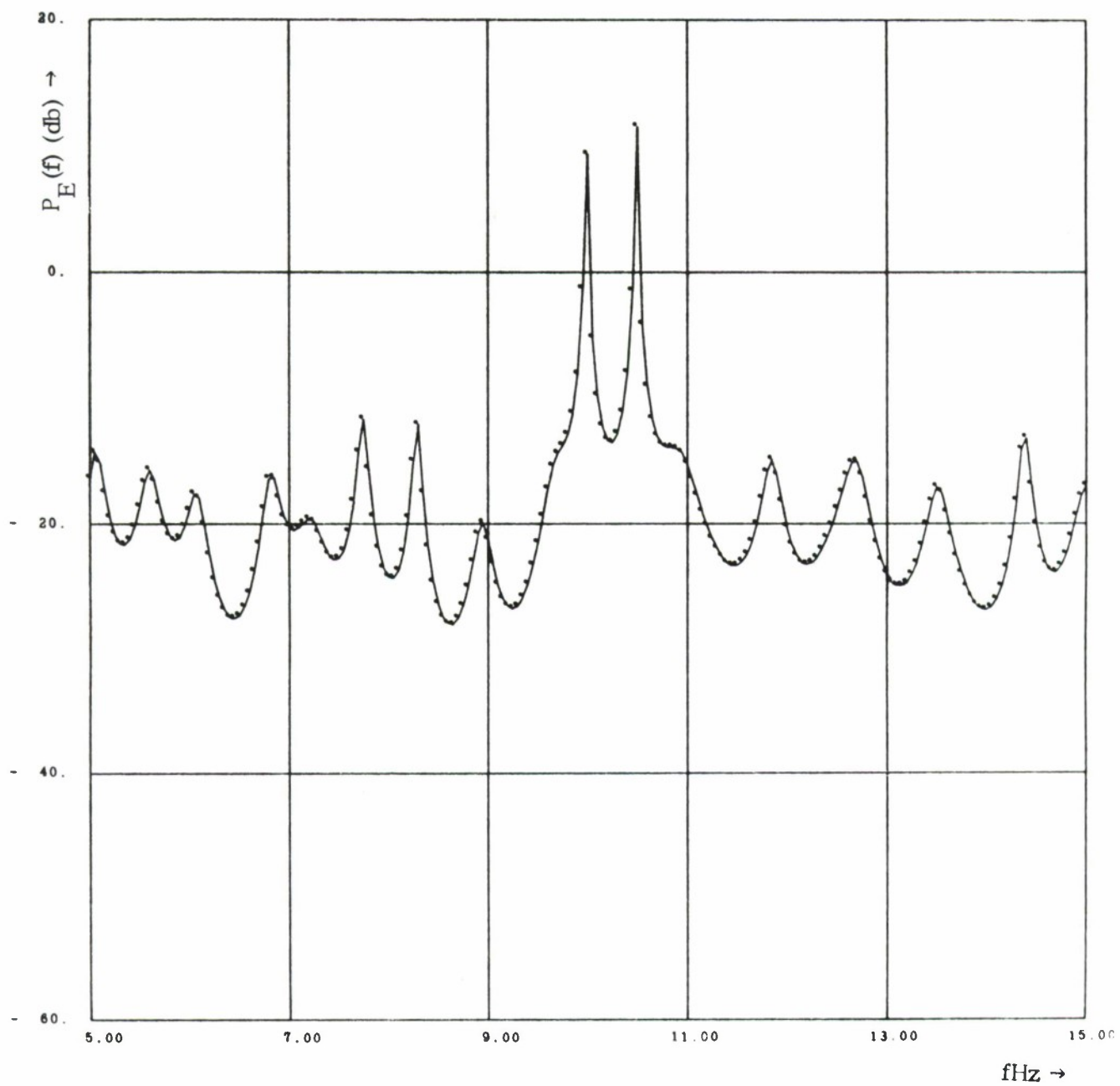
Fig. 5.23 - Direct-Hanning, ρ = .4.
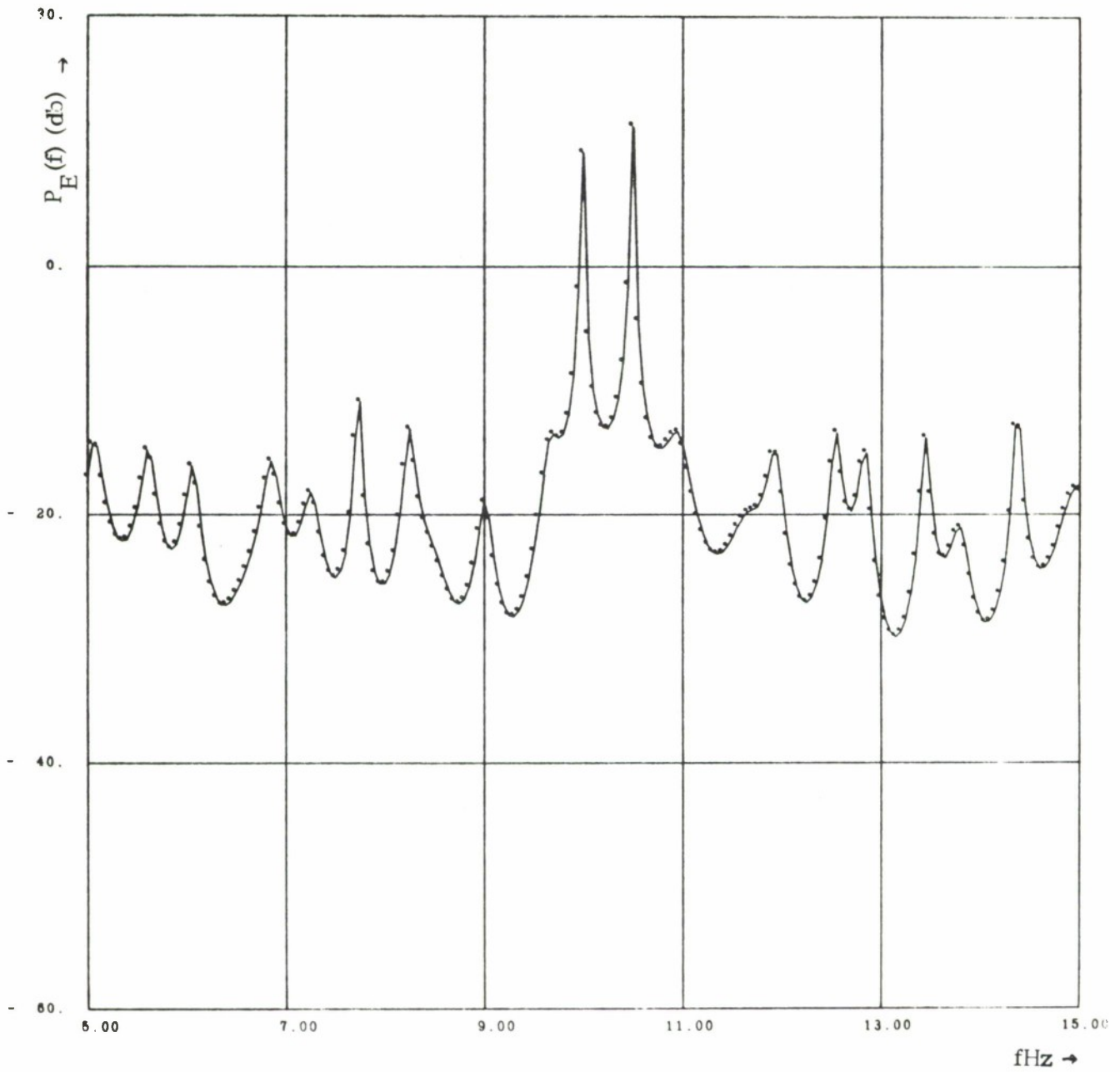
Fig. 5.24 - Direct-Hanning, ρ = .5.
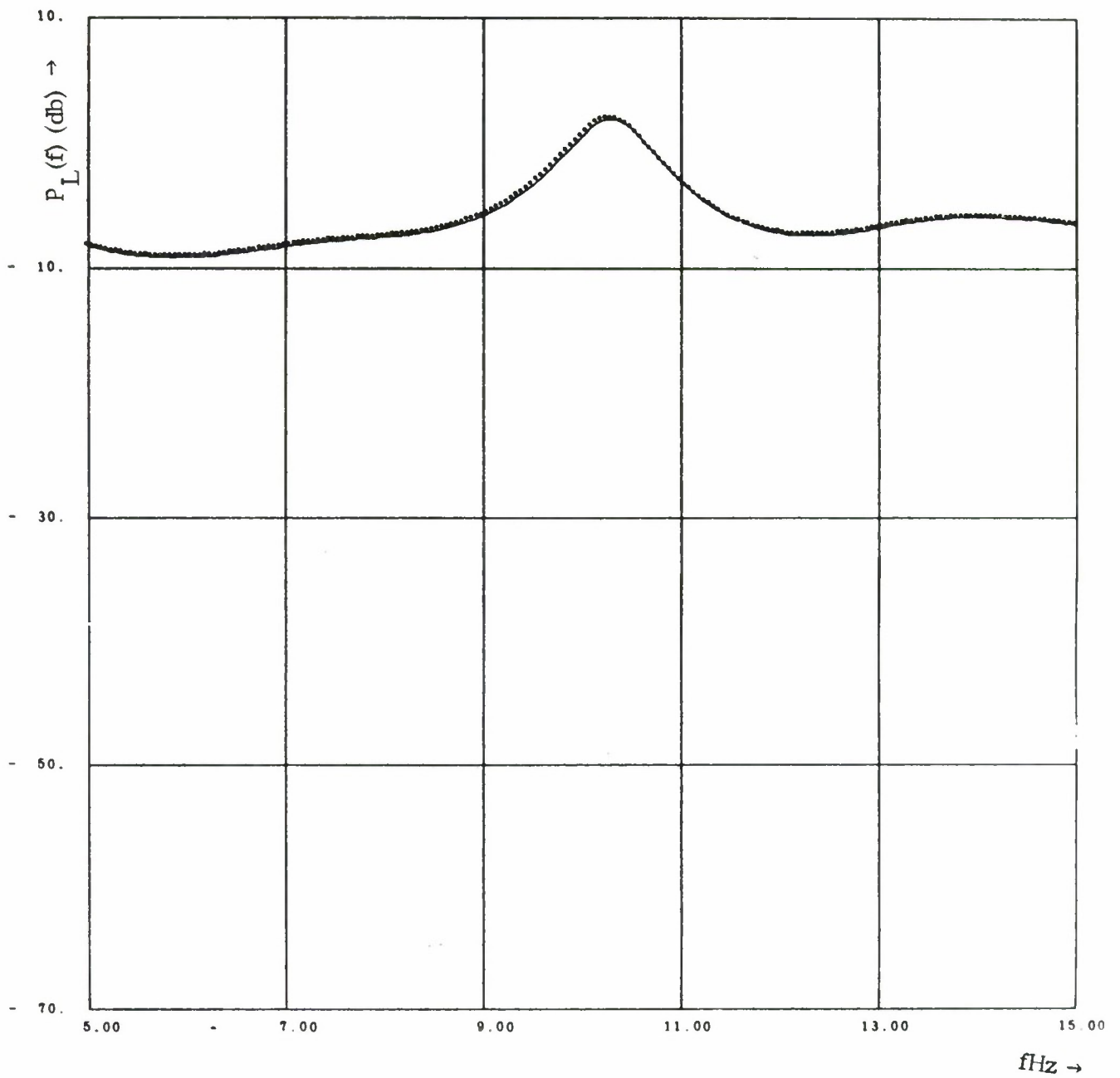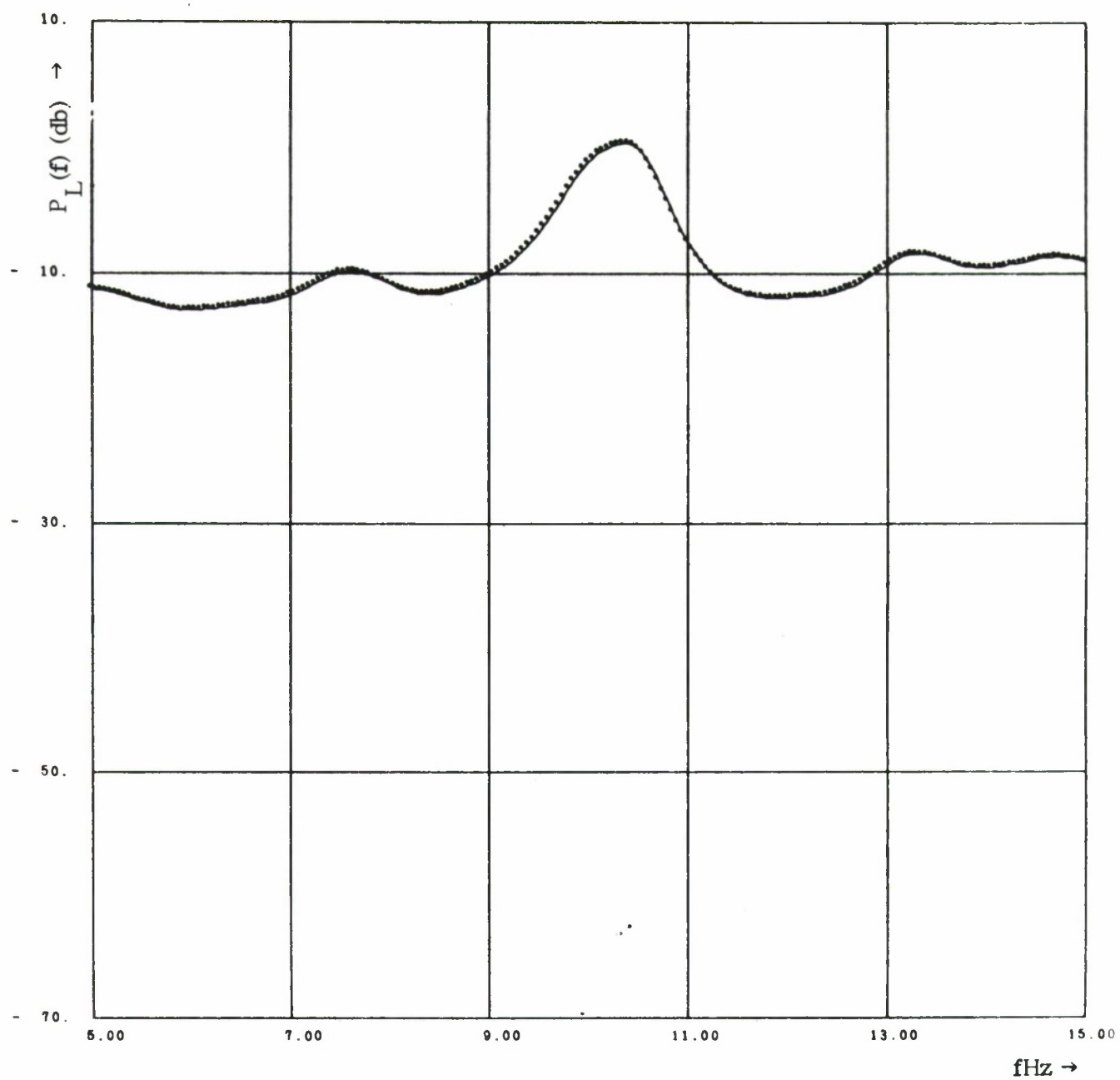
Fig. 5.25 - Indirect, ρ = .1.

Fig. 5.26 - Direct, $\rho = .1$.

Fig. 5.27 - Indirect, ρ = .2.
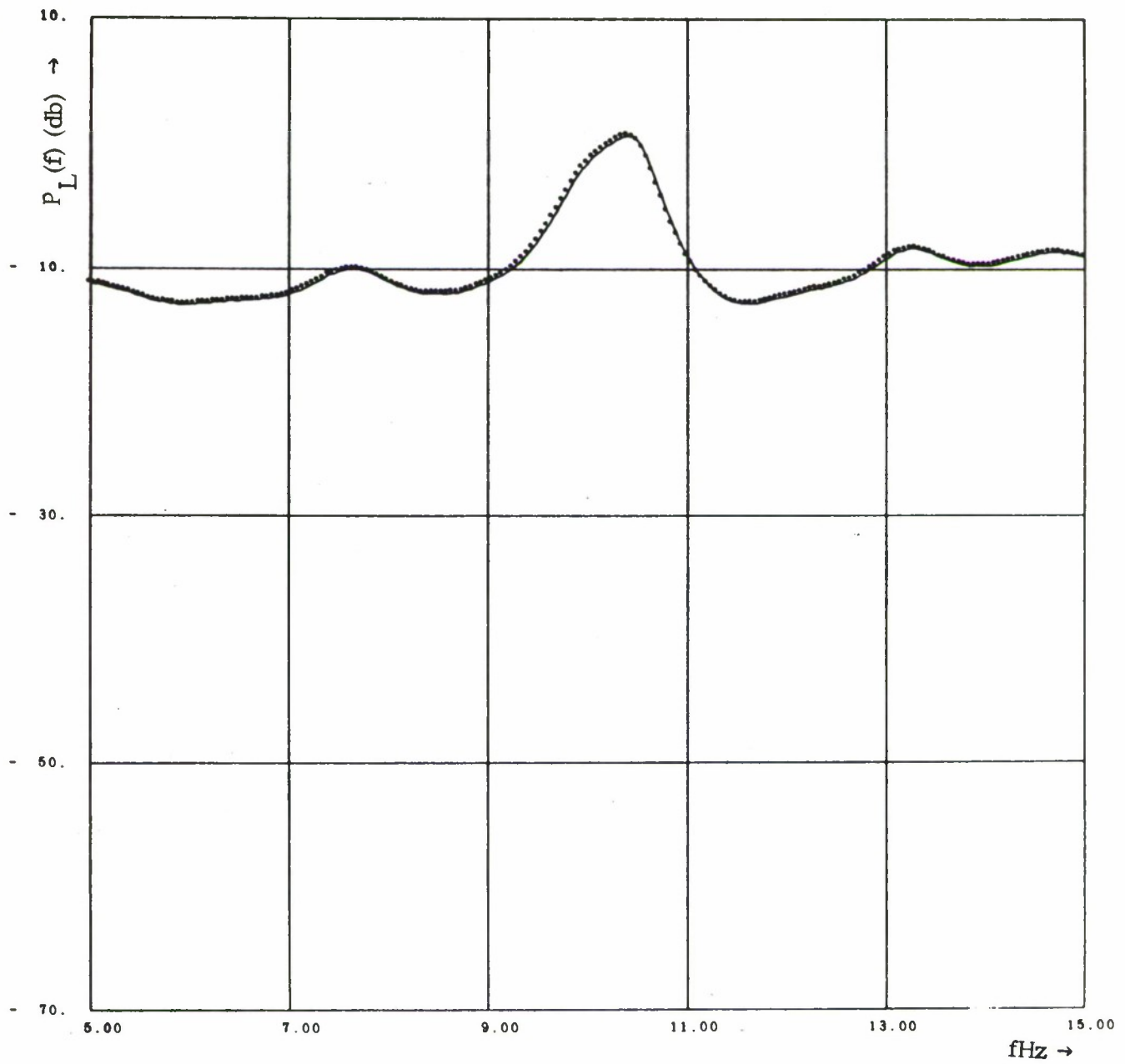
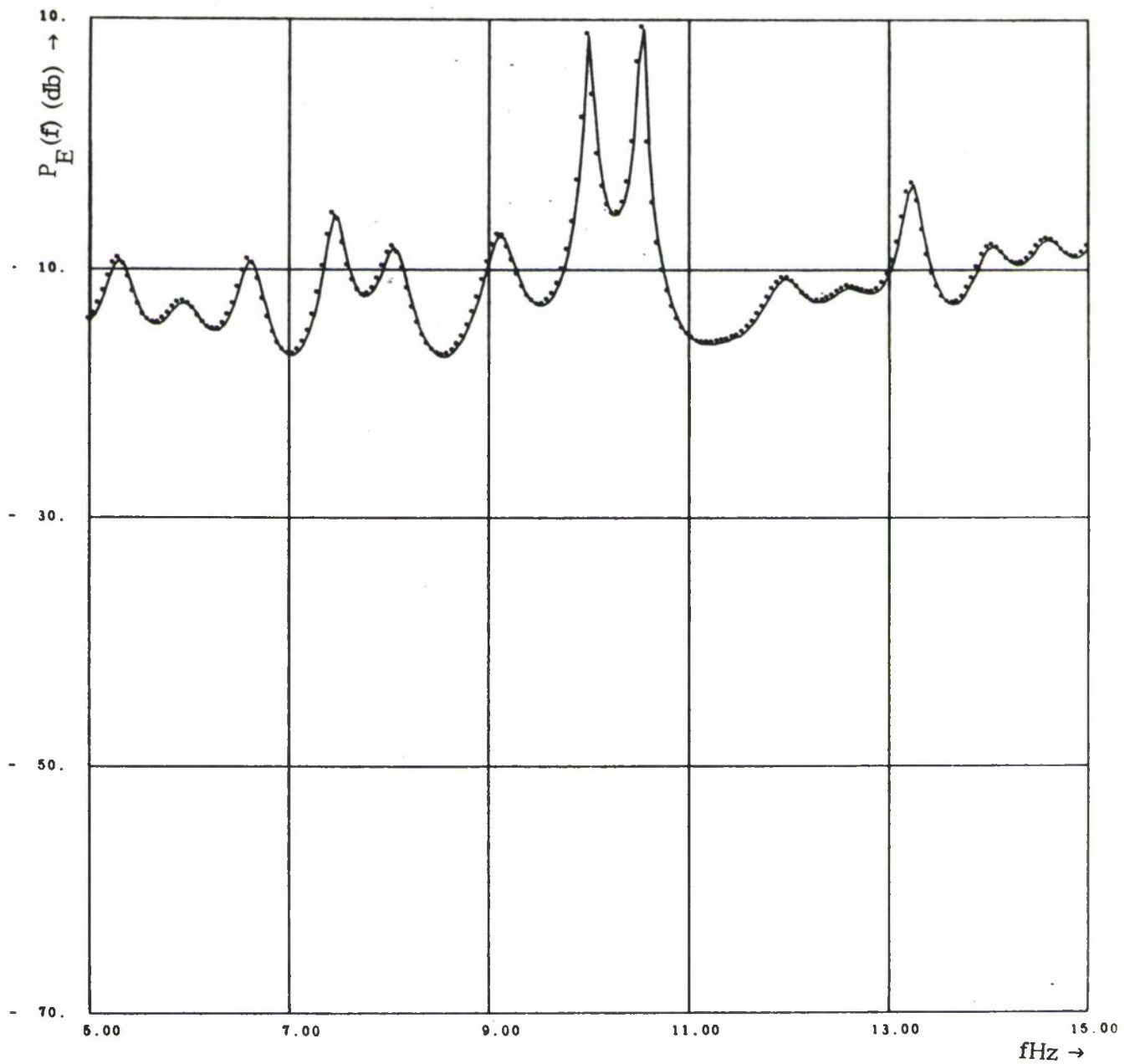Fig. 5.28 - Direct, ρ = .2.
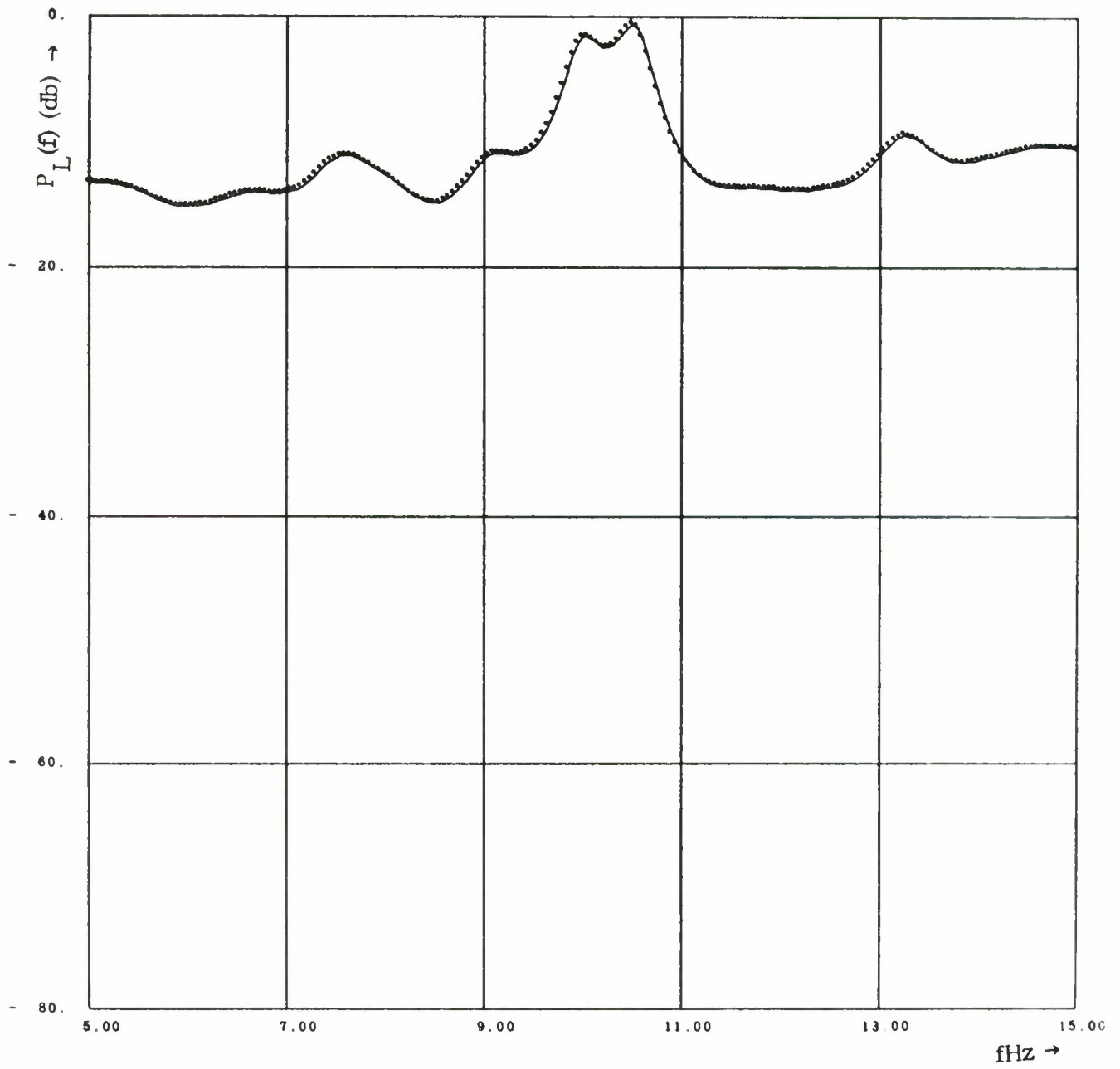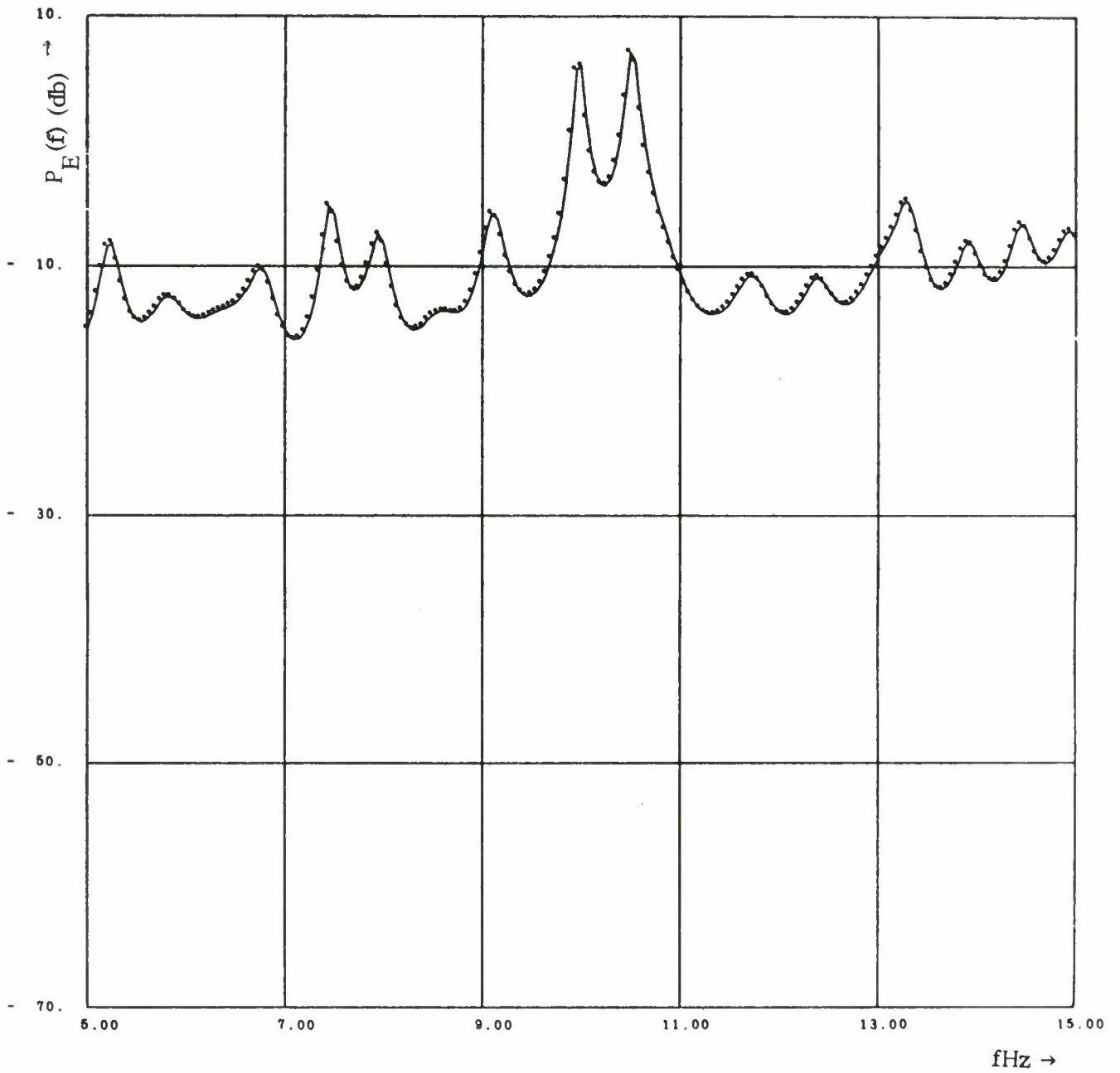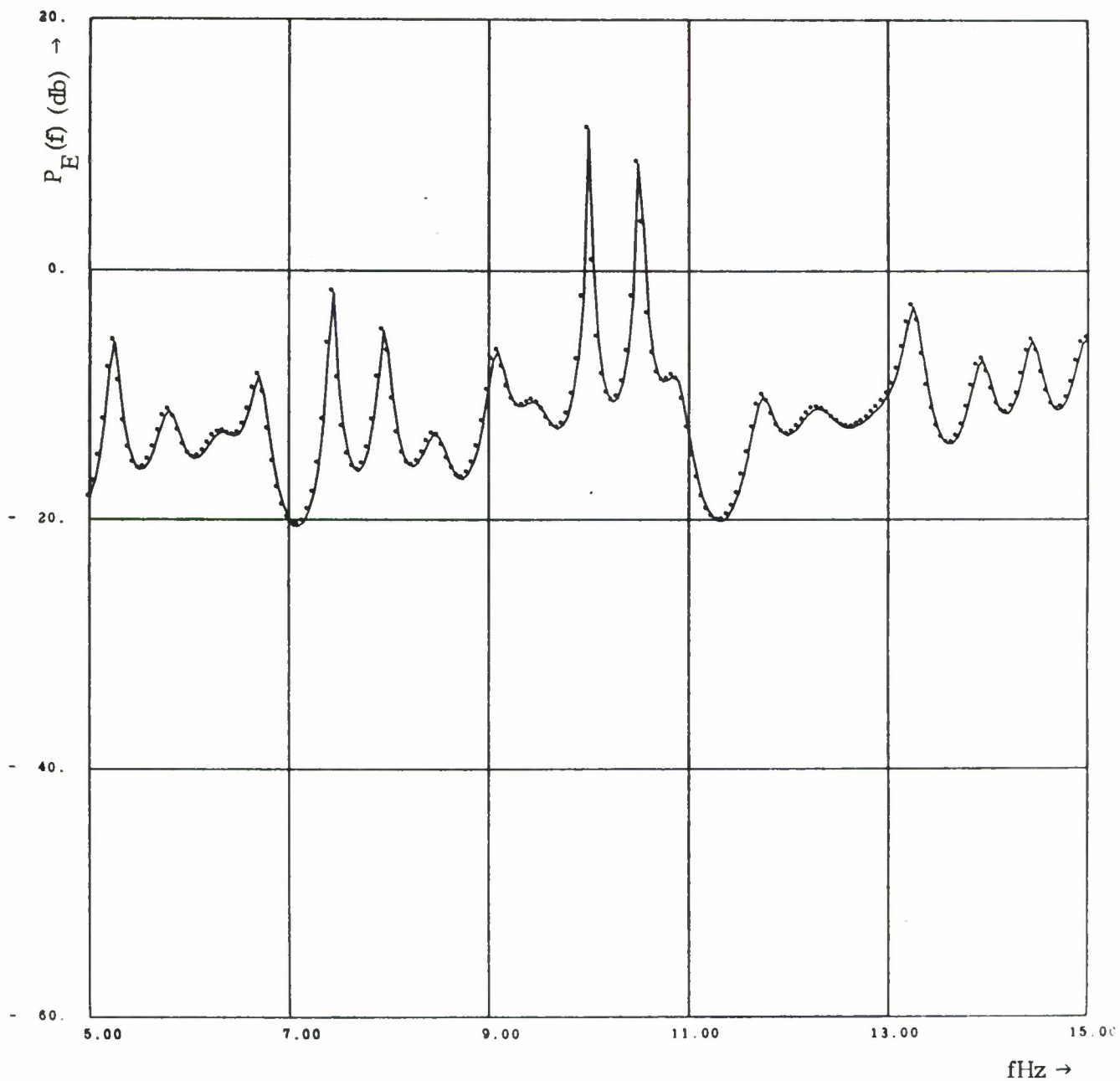
Fig. 5.29 - Indirect, ρ = .3.
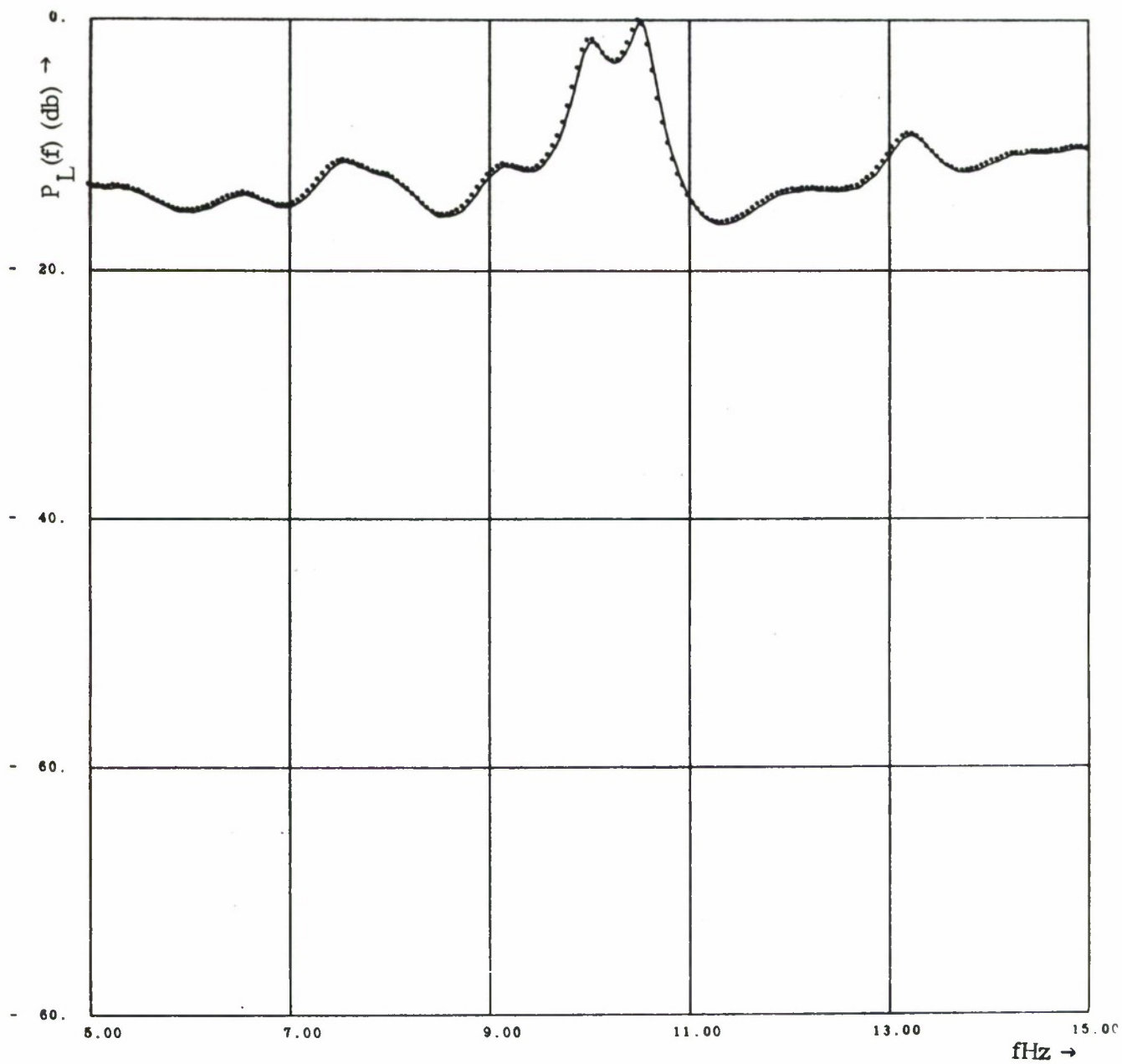
Fig. 5.30 - Direct, ρ = .3.

Fig. 5.31 - Indirect, $\rho = .4$.

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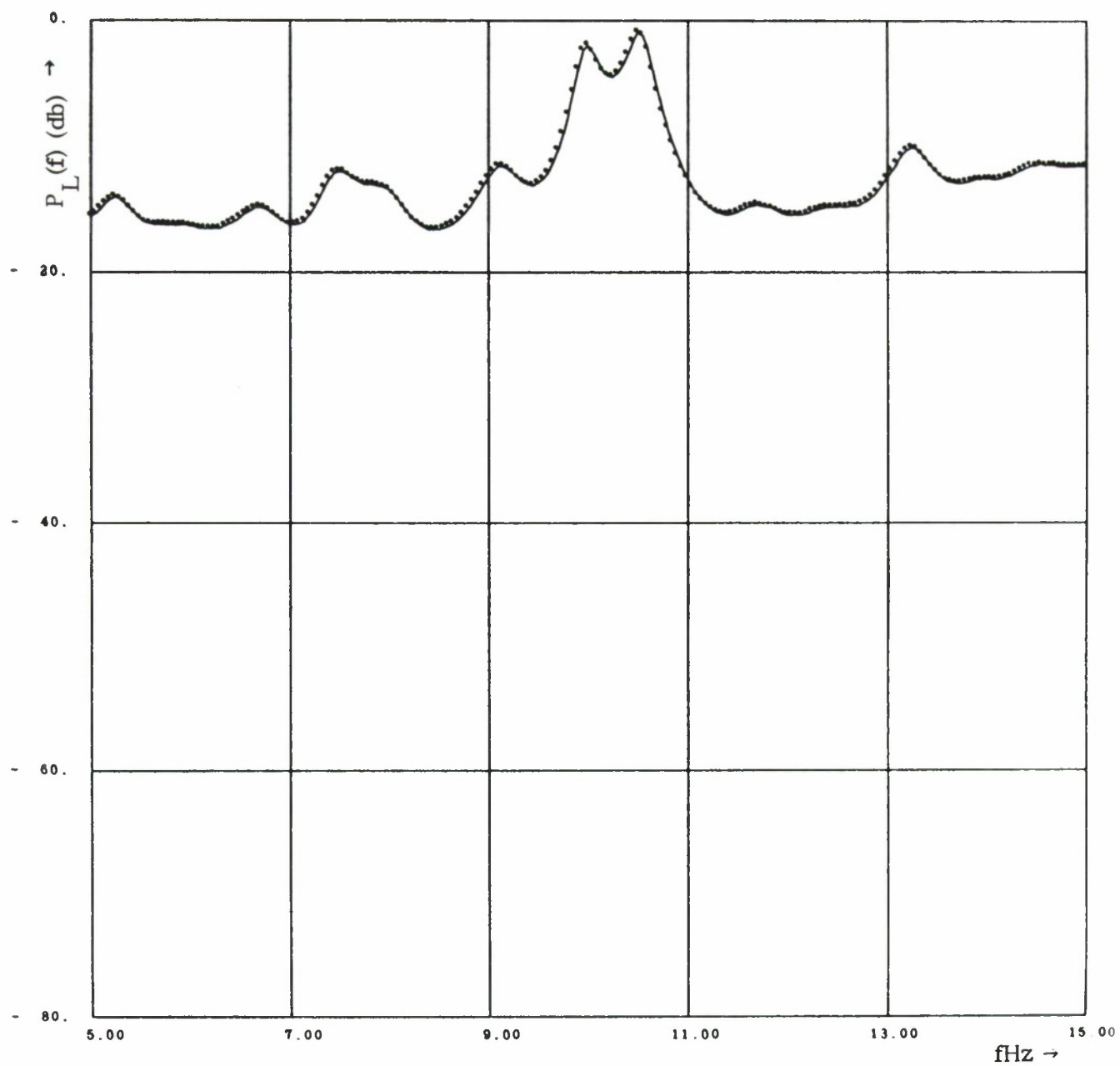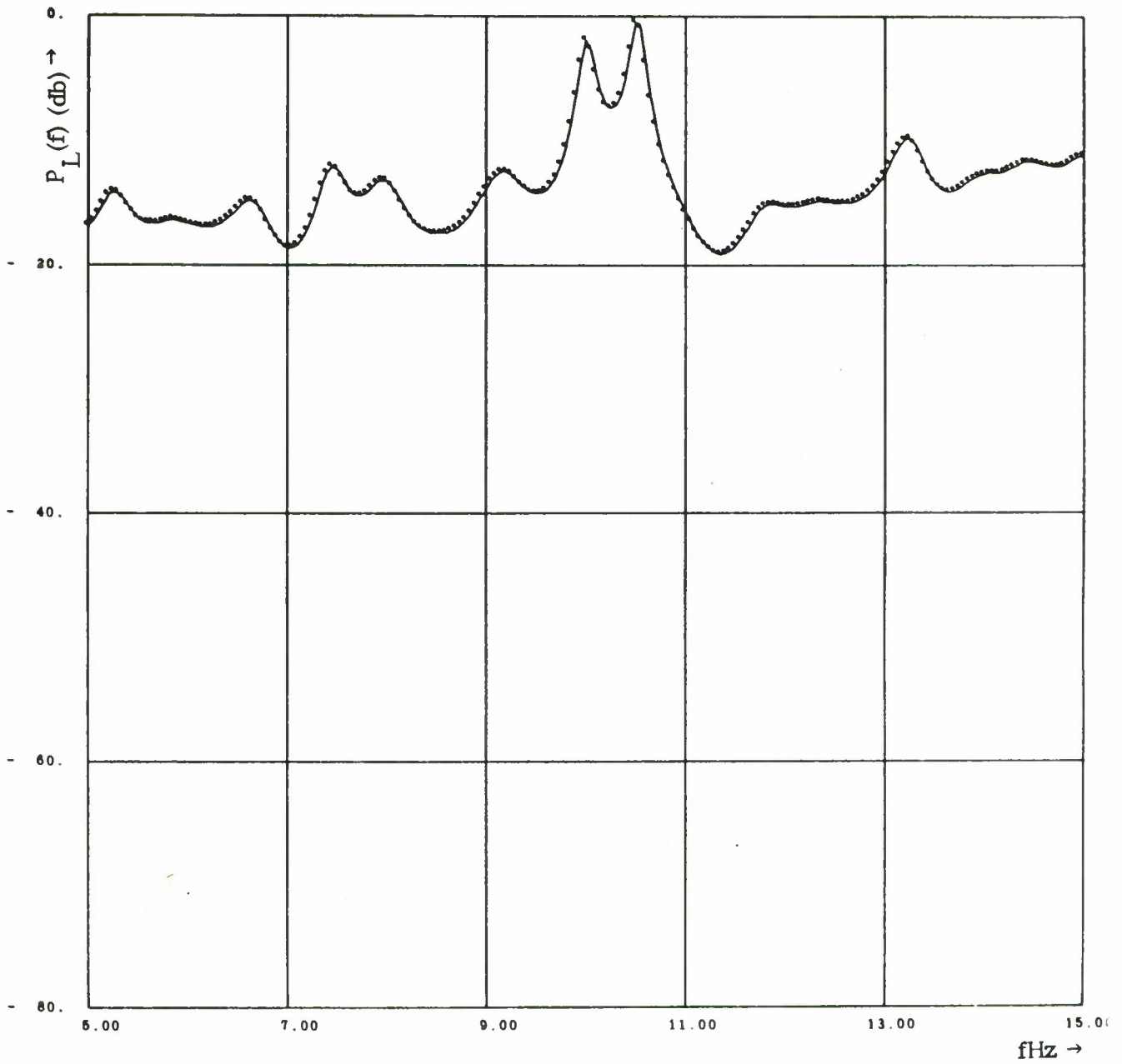
Fig. 5.32 - Direct, ρ = .4.
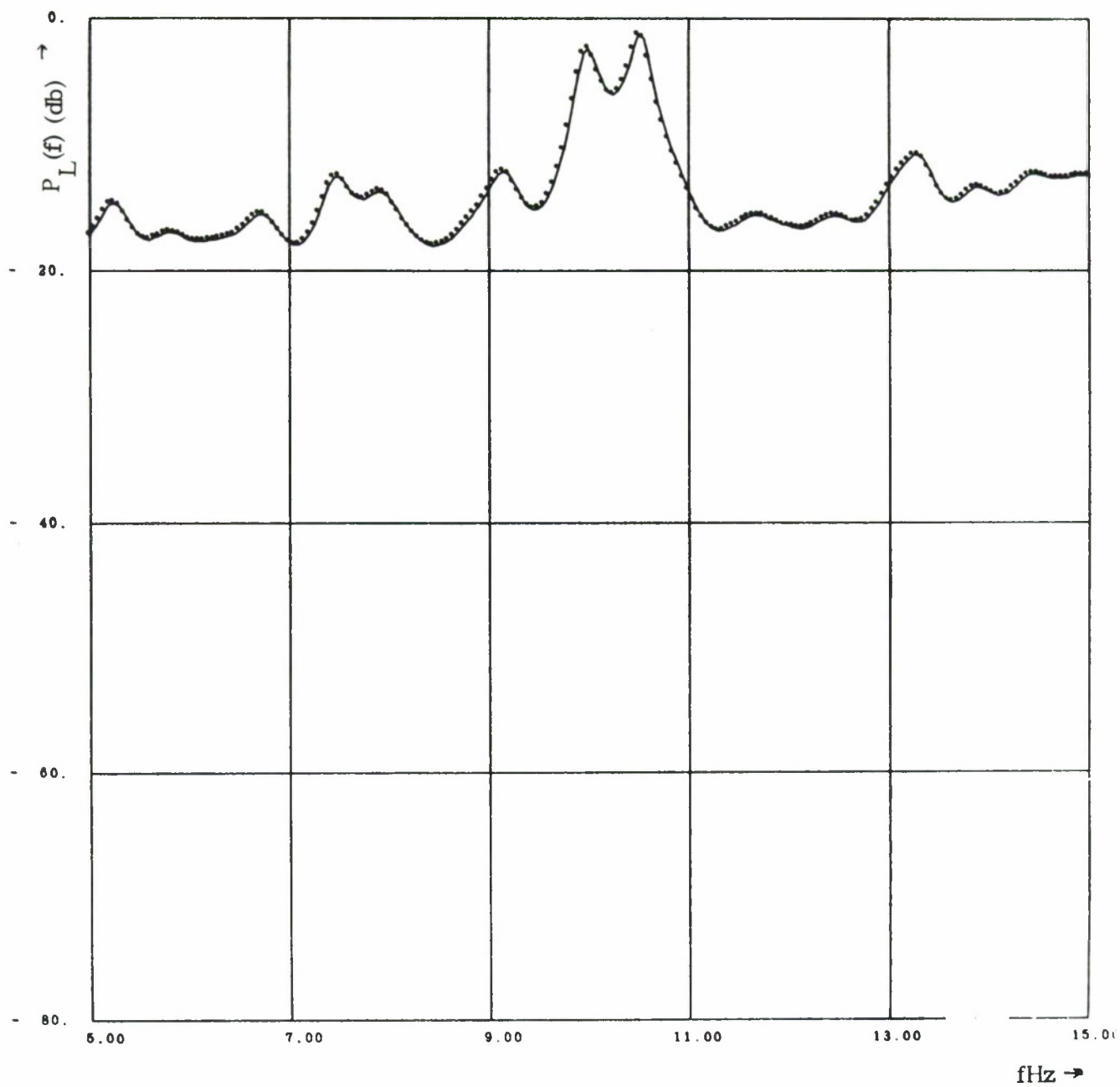
Fig. 5. 33 - Indirect, $\rho = .5$.

Fig. 5.34 - Direct, ρ = .5.

Fig. 5.35 - Direct-Hanning, ρ = .4.

Fig. 5. 36 - Direct-Hanning, $\rho = .5$.

Fig. 5.37 - Indirect, $\rho = .1$.

Fig. 5.38 - Direct, $\rho = .1$.

Fig. 5.39 - Indirect, ρ = .2.
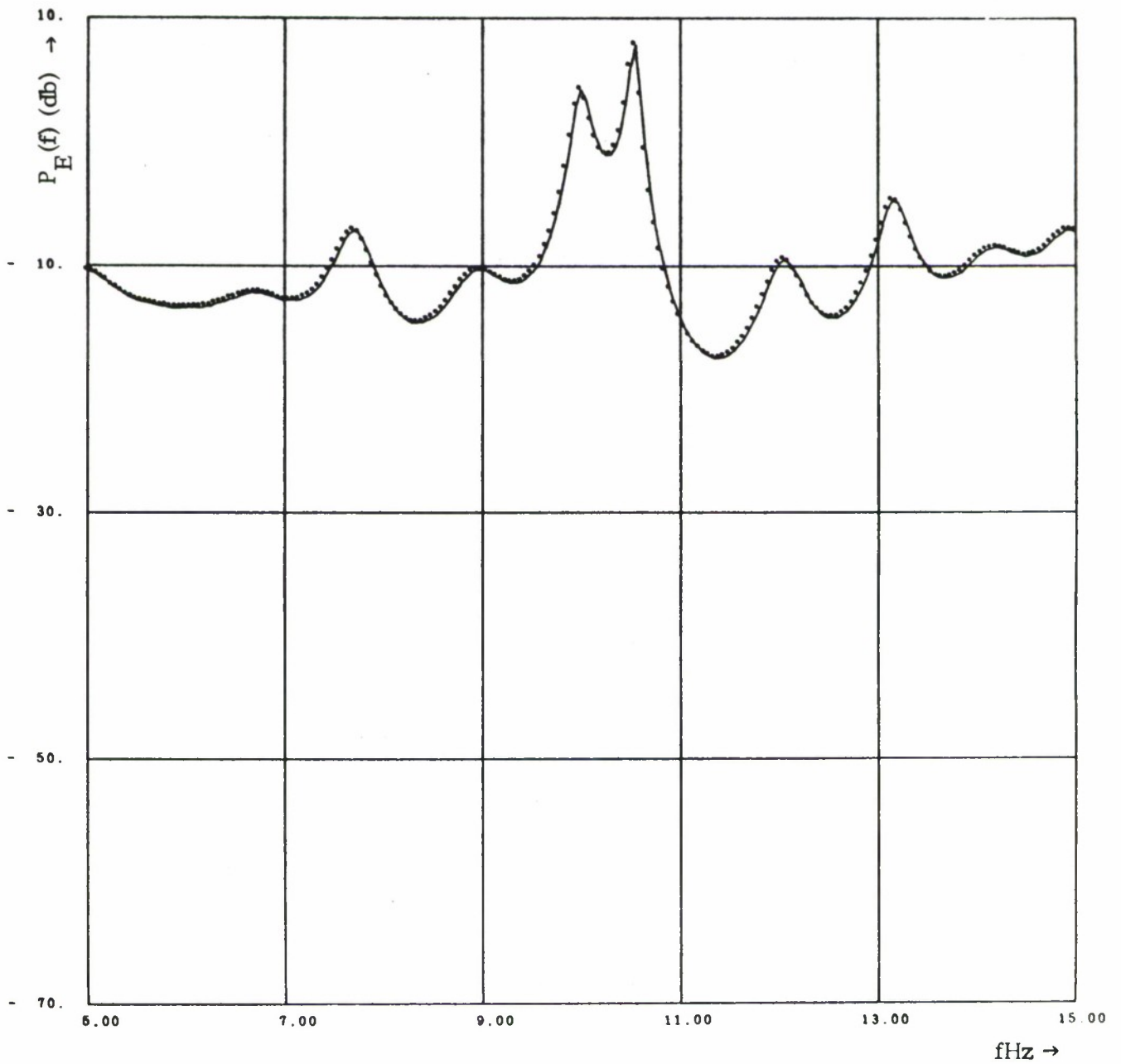
Fig. 5.40 - Direct, ρ = .2.

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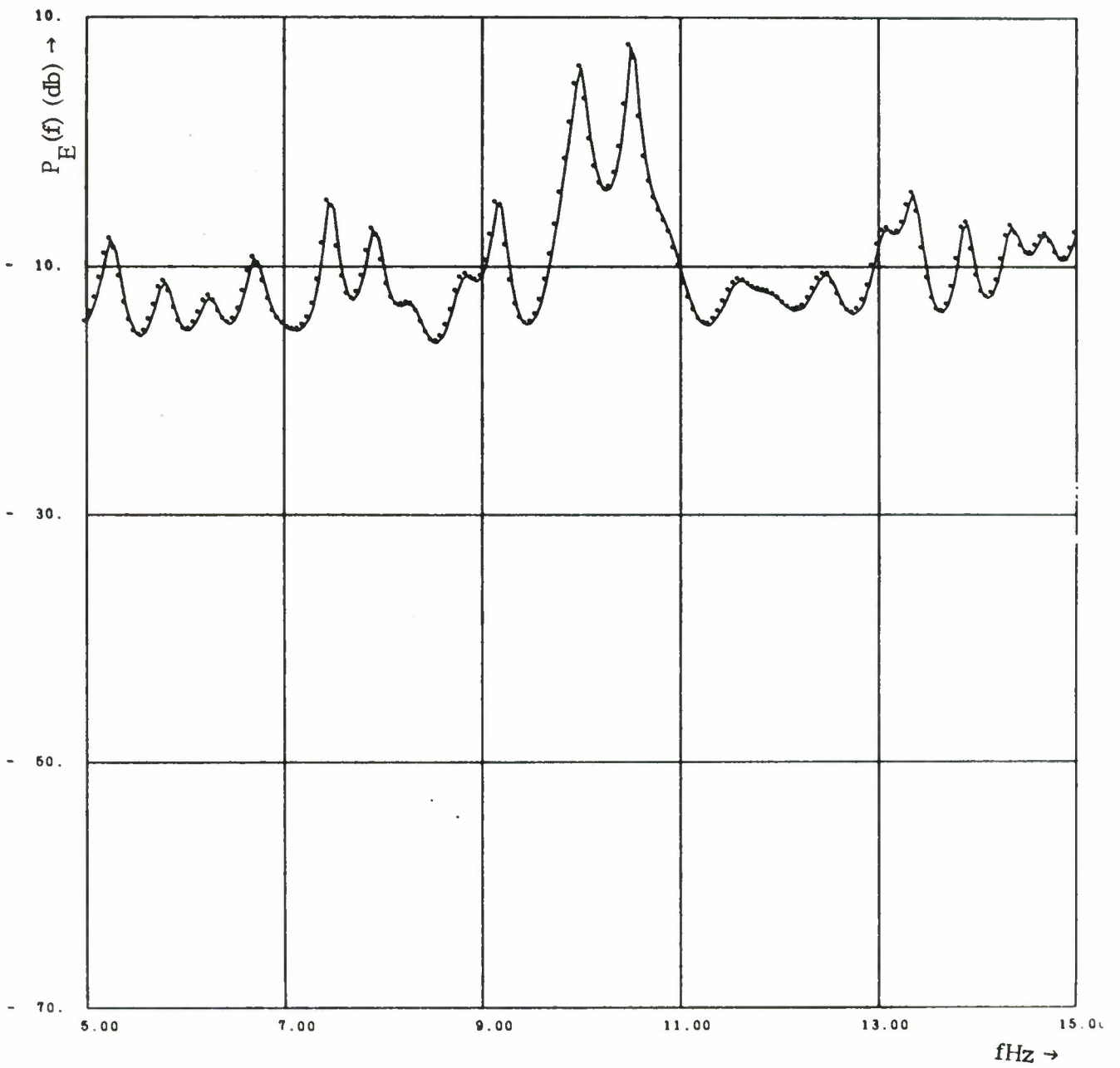Fig. 5.41 - Indirect, ρ = .3.

Fig. 5.42 - Direct, ρ = .3.

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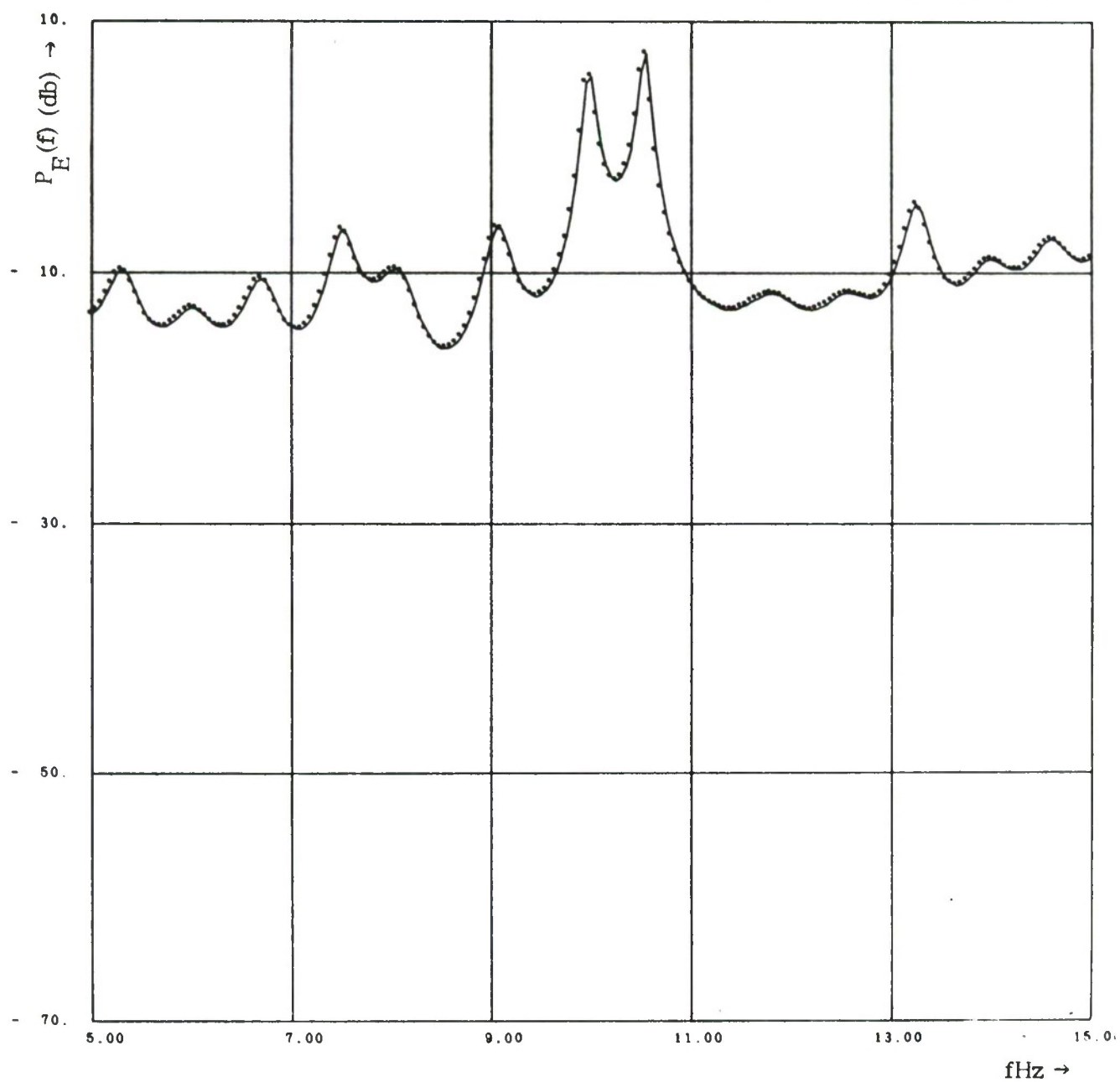
Fig. 5.43 - Indirect, ρ = .4.

Fig. 5.44 - Direct, $\rho = .4$.
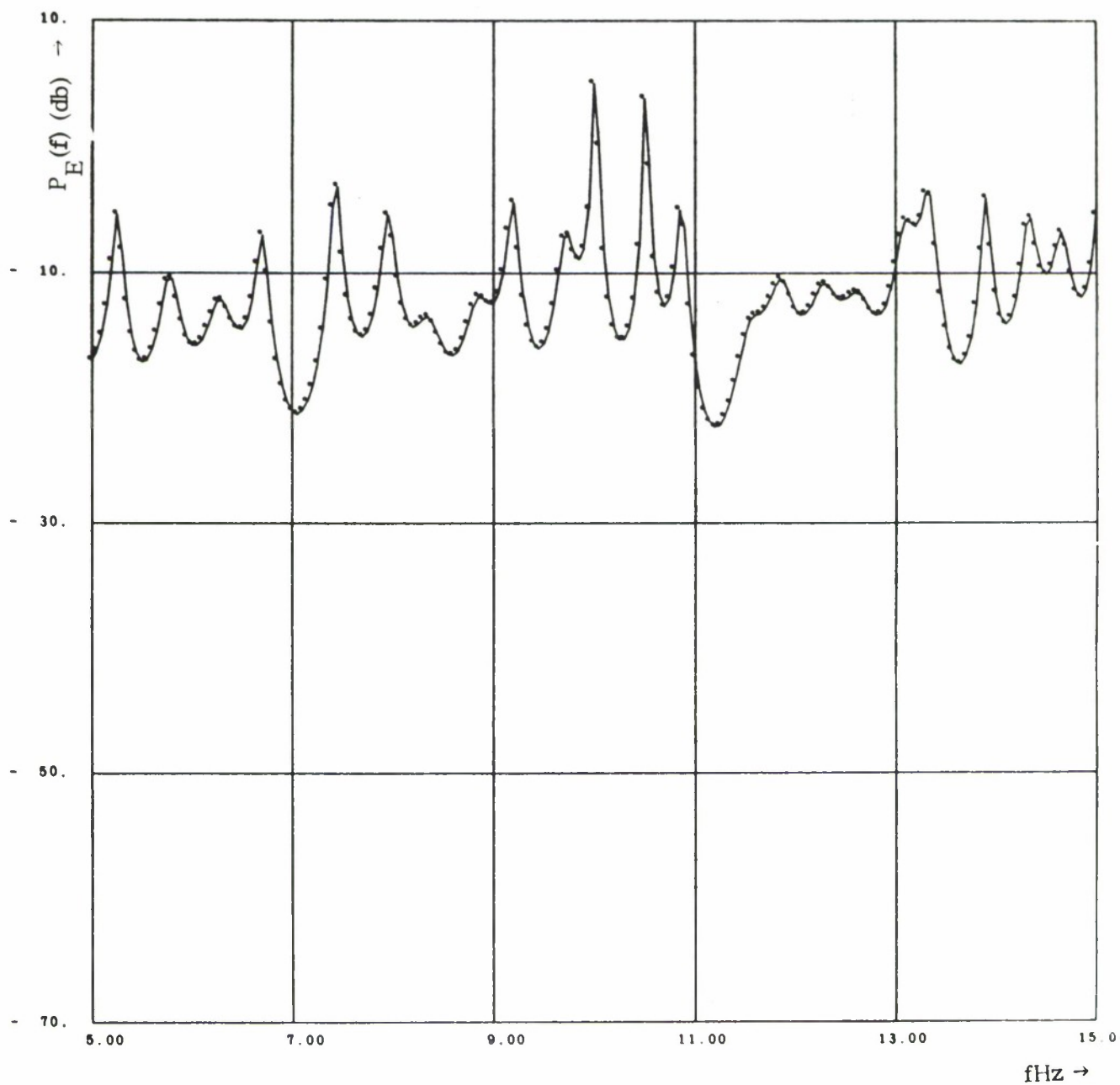
Fig. 5.45- Indirect, $\rho = .5$.
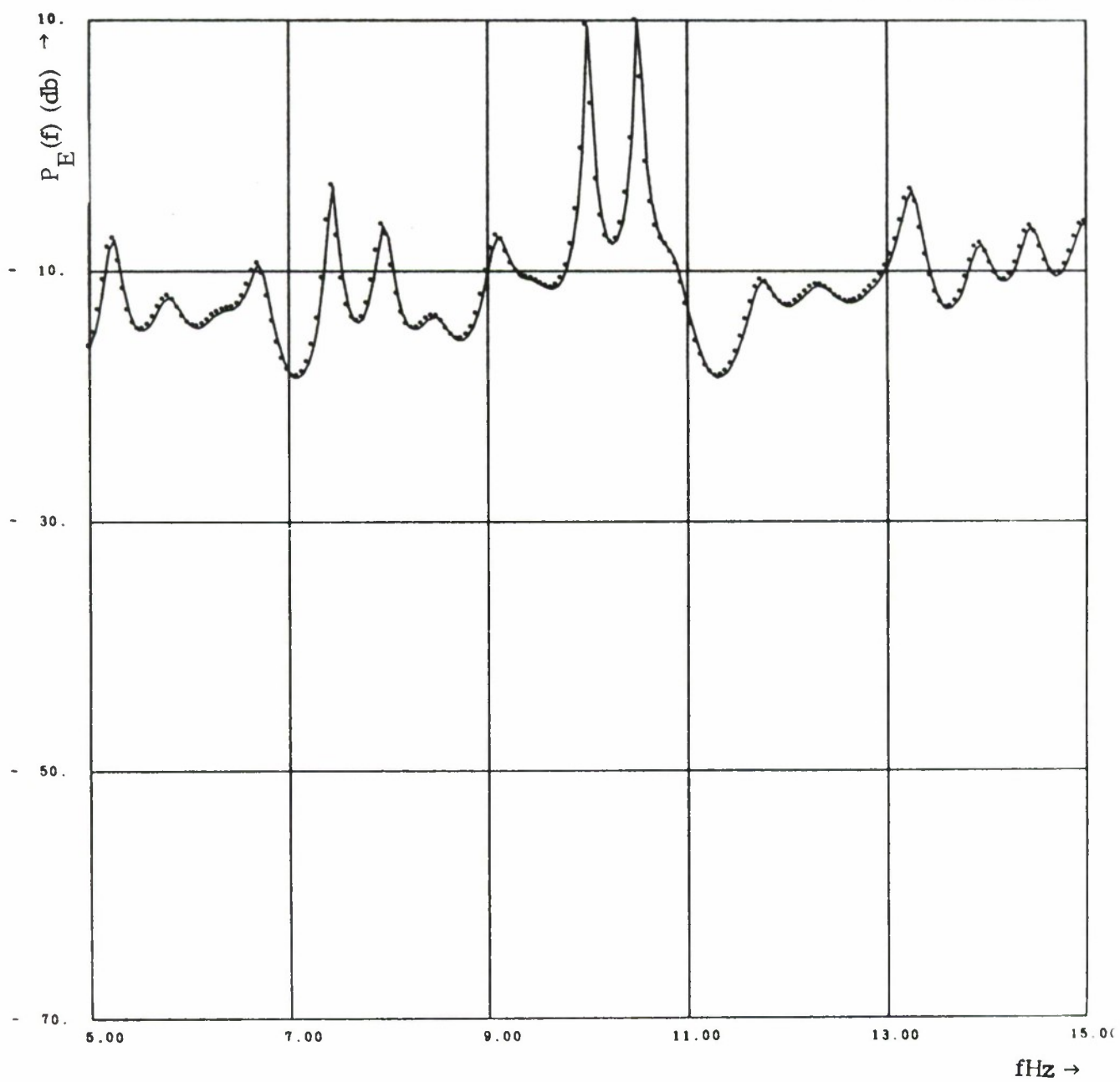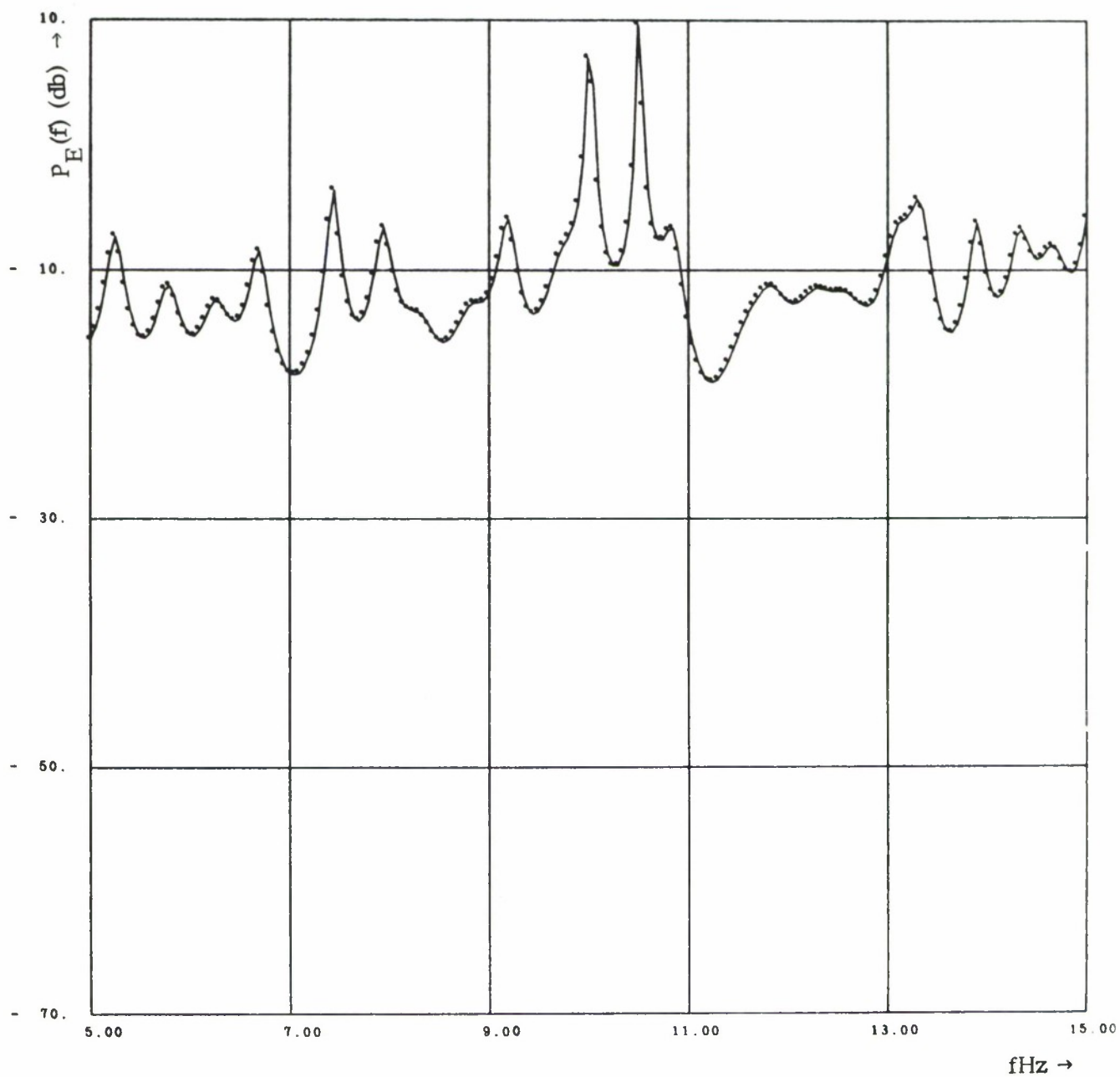
Fig. 5.46 - Direct, ρ = .5.

Fig. 5.47 - Direct-Hanning, $\rho = .4$.

Fig. 5.48 - Direct-Hanning, ρ = .5.

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The Maximum Likelihood Method and the Maximum Entropy Method of spectral estimation are described and interpreted in terms of the innovation filter concept. Algorithms are developed for mapping an observed data sequence into the spectral estimates. The resolution performance of these spectral estimation algorithms is compared by applying each to simulated data containing two tones in white noise.

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